

Department of Physics and Astronomy  
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Bachelor Thesis in Physics  
submitted by

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**2018**

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Date: September 17th, 2018

# Effective theory of Goldstone bosons in an $N$ component Gross-Pitaevskii model



## Abstract

The behavior of a quantum field theory exhibiting spontaneous symmetry breaking (SSB) is dominated at low energies by massless degrees of freedom. These Goldstone bosons can be predicted in number and behavior by Goldstone's theorem. The theorem allows to derive an effective field theory where Goldstone bosons are the only degrees of freedom.

We study spontaneous symmetry breaking in a dilute Bose gas which is governed by the Gross-Pitaevskii (GP) model with  $N$  components and  $O(N)$  symmetric interactions. The model and the phenomenon of spontaneous symmetry breaking are discussed shortly. It is shown that Goldstone bosons in the Gross-Pitaevskii model arise from a  $U(N) \rightarrow U(N-1)$  type symmetry breaking. An appropriate parametrization of the quotient  $U(N)/U(N-1)$  allows for a description of the theory in one heavy and  $2N-1$  massless quasiparticles. With this it can be observed that  $2N-2$  modes with the dispersion relation of a non relativistic free particle arise besides the well known Bogoliubov mode. Parallel steps to recent research [1] are taken from here. A low energy effective theory can be computed by using only one integration of a longitudinal mode without further requirements on the chosen ground state. Relative density fluctuations on the other hand can still be strongly excited giving a complementary picture to [1]. Motivated by the works of Watanabe and Murayama [2], it is demonstrated that the free particle modes pairwise describe one degree of freedom, answering to a previously open question. The interaction of the modes is studied using the effective theory. Here no interaction of three free particle modes can be observed. Finally some work towards a kinetic description is done revealing technical difficulties with the approach in the case of large field fluctuations. Moreover the non triviality of arising homotopy groups is pointed out, possibly hinting at the existence of topological charges in the case  $d=3, N=2$ .

## Zusammenfassung

Das Verhalten einer Quantenfeldtheorie, die spontane Symmetriebrechung aufweist, wird bei geringen Energien maßgeblich durch massenlose Anregungen bestimmt. Diese Goldstone Bosonen können in ihrer Zahl und in ihrem Verhalten durch das Goldstone Theorem vorhergesagt werden. Das Theorem erlaubt es zu einer effektiven Feldtheorie überzugehen in der nur noch massenlose Freiheitsgrade auftreten.

Wir studieren die spontane Symmetriebrechung in einem Bose Gas geringer Dichte, welches durch das Gross-Pitaevskii Modell mit  $N$  Komponenten und  $O(N)$  Symmetrie beschrieben werden kann. Das Modell und das Phänomen spontaner Symmetriebrechung werden kurz vorgestellt. Es wird gezeigt, dass den auftretenden Goldstone Bosonen eine Symmetriebrechung  $U(N) \rightarrow U(N-1)$  zu Grunde liegt. Eine geeignete Parametrisierung des Quotienten  $U(N)/U(N-1)$  ermöglicht es die Theorie durch eine massive und  $2N-1$  massenlose Freiheitsgrade zu beschreiben. Dabei treten neben der aus der Literatur bekannten Bogoliubov Mode  $2N-2$  weitere Moden mit der Dispersionsrelation eines freien nichtrelativistischen Teilchens auf. Es wird nun parallel zu einer früheren Arbeit [1] vorgegangen. Durch lediglich eine Integration über den longitudinalen Freiheitsgrad ohne Annahmen an den angenommenen Grundzustand kann zu einer effektiven Theorie übergegangen werden. Hierbei können relative Dichtefluktuationen stark angeregt sein, weshalb der Ansatz ein komplementäres Bild zu [1] liefert. Motiviert durch die Arbeit von Watanabe und Murayama [2] lässt sich zeigen, dass die freien Moden jeweils paarweise einen Freiheitsgrad beschreiben womit eine zuvor ungeklärte Fragestellung beantwortet wird. Die Wechselwirkung der Moden wird mit der effektiven Theorie untersucht. Dabei tritt keine Wechselwirkung von drei Moden mit freier Dispersionsrelation auf. Abschließend werden Zwischenresultate für eine kinetische Theorie dargestellt, wobei Schwierigkeiten des Ansatzes insbesondere im Fall starker Feldfluktuationen deutlich werden. Außerdem wird auf die Nichttrivialität auftretender Homotopiegruppen hingewiesen, die auf die Existenz topologischer Ladungen im System im Fall  $d=3, N=2$  hindeuten.



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# 1. Introduction

## 1.1. Overview

Bose-Einstein condensation describes a state of matter in which bosons condense in the lowest lying energy state of a system. The phenomenon was first described over 90 years ago [3] and it took 70 years until a Bose-Einstein condensate was first realized experimentally. Since then Bose-Einstein condensates have gained a lot of attention by researches and it turned out to be an especially helpful tool to study quantum physics. One reason for this is the prominent role of symmetries in modern physics. Symmetries can lead to very similar behavior in otherwise completely different systems. This is called universality and usually appears in the vicinity of very specific „critical“ points, for which the most famous one is the point of a phase transition. However, also far from equilibrium universality can appear, where the evolution equations are governed by just a few parameters in a scaling law. The theoretical understanding of such a non-thermal fixed point in a Bose-Einstein condensate is of particular importance, as condensates are experimentally accessible nowadays.

My work consists mainly in deriving a low-energy effective description of the Gross-Pitaevskii model for Bose-Einstein condensation to study the number and interactions of massless degrees of freedom in the system. These degrees of freedom dominate the low energy behavior of the model and hence are important to investigate non-thermal fixed points. The Gross-Pitaevskii model is given by

$$\mathcal{L} = \frac{i}{2}(\varphi_a^*(\partial_t \varphi_a) - \varphi_a(\partial_t \varphi_a^*)) - \frac{1}{2m}(\partial_x \varphi_a^*)(\partial_x \varphi_a) - \frac{g}{2}(\varphi_a^* \varphi_a)^2 + \mu \varphi_a^* \varphi_a. \quad (1.1)$$

It describes the condensate of a bosonic dilute gas with  $N$  distinguishable components, which can for example represent different magnetic sublevels of a given atom. The components are not interacting but of the same particle type, such that transitions from one component to another are possible. In a recent development members of our group used the approach to represent the fields in terms of phases and densities

$$\varphi_a(x) = \sqrt{\rho_a(x)} e^{i\theta_a(x)}, \quad a = 1 \dots N, \quad x \in \mathbb{R}^{d+1} \quad (1.2)$$

to describe the physics at a non-thermal fixed point in a perturbative way [1]. This is done by integrating out the density degrees of freedom  $\rho_a$  at a Gaussian level in the pathintegral. With this one obtains an effective theory in  $N$  massless collective phase excitations that drive the dynamics of the condensate. The total phase  $\theta_B = \sum_{a=1}^N \theta_a$  obeys a Bogoliubov dispersion

$$\omega_B(\mathbf{k}) = \sqrt{\frac{\mathbf{k}^2}{2m} \left( \frac{\mathbf{k}^2}{2m} + 2g\rho^{(0)} \right)} \quad (1.3)$$

which is well known from the one dimensional Gross-Pitaevskii model, while the relative phases  $\theta_G = \theta_1 - \theta_a$  keep the free particle dispersion

$$\omega_G(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} \quad (1.4)$$

of the fundamental fields. The appearance of these modes is no surprise. The fundamental reason for them is a process of spontaneous symmetry breaking when the gas condenses in the lowest lying energy state. This symmetry breaking is then connected via Goldstones theorem to a specific number of bosonic quasiparticles which are also called Goldstone modes or Goldstone bosons. In fact, the theorem predicts for our model  $2N - 1$  Goldstone bosons, while so far we discussed only  $N$  phase excitations. However, Goldstones theorem is a priori only applicable to relativistic theories. This will come to our rescue. In the following work I want to discuss Goldstones theorem in the context of a non-relativistic theory. The study of the symmetry breaking does not only help to clarify why

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we are missing modes. It also provides with an alternative parametrization of the fundamental fields which might allow for a more detailed study of the modes' interactions. In addition the mathematical concepts are accessible and geometrical, possibly allowing for an application of this idea also to more complicated cases. I first want to discuss Goldstones theorem in the general context before making the transition to the non relativistic case. It is possible to formulate an effective theory of Goldstone bosons by integrating out the longitudinal degree of freedom. We will see that modes pairwise describe one degree of freedom. Additionally, propagators and interaction vertices between different modes can be determined and topological properties can be studied.

### 1.2. Spontaneous symmetry breaking

If we consider a physical system with a certain symmetry, say rotational symmetry, then the lowest lying energy state might not respect that symmetry. In this case the system can exhibit spontaneous symmetry breaking (SSB) at low energies. The phenomenon can be illustrated with the Heisenberg model of ferromagnetism. Imagine a lattice of  $N$  ferromagnetic spins, which means that two aligned spins are energetically favorable. The model can be described by the Hamiltonian  $H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \mathbf{S}_j$  where the sum runs over neighbored spins. [4]. In three dimensions this Hamiltonian is clearly rotationally invariant because interaction have an  $O(3)$  symmetry. However, the lowest lying energy state corresponds to a configuration where all spins are aligned. This alignment could point in any direction, and the configuration isn't invariant under rotations that change the axis of alignment. The groundstate has only an  $O(2)$  symmetry.

The ordered and disordered phase are distinguished by the associated magnetization. In the disordered state the magnetization vanishes,  $\mathbf{m} = 0$ , while the magnetization becomes non trivial in the groundstate. It serves as a so called order parameter. The transition between the two phases can be understood by bringing the system in contact with a heat bath. This makes the temperature an adjustable parameter and we have a minimization principle for the free energy  $F = \langle H \rangle - TS$ . At high temperatures, entropy dominates  $F$  and we are in a disordered state with rotational symmetry, which maximizes the entropy. If temperature falls below a critical value energy takes over and we land in a configuration with aligned spins, which minimizes  $\langle H \rangle$ . The direction of alignment is picked spontaneously. We just observed spontaneous symmetry breaking. Note that it is a non trivial question whether the transition between ordered phase and disordered phase actually happens at  $T \neq 0$ .

Symmetries play a great role in physics, and consequently spontaneous symmetry breaking offers the possibility for interesting phenomena. The situation we just considered allows for example for two transversal modes in the ordered phase. "Shaking" one spin affects neighbored spins and yields the propagation of a wave in the magnet. This transversal wave is a first example of Goldstone bosons, although in a semi-classical picture. For the treatment of the Gross-Pitaevskii model we need a field theoretic approach.

#### 1.2.1. Noether's theorem

First we will have a look at Noether's theorem, a fundamental theorem that connects continuous symmetries and conserved currents. This discussion is inspired by [5]. We consider a transformation  $g$  that acts on fields  $\varphi_a$  via  $\varphi'_a = g_{ab}\varphi_b$ . In this context we speak of a symmetry if the action  $S[\varphi_a, \varphi_a^*, \partial\varphi_a, \dots] = \int \mathcal{L}[\varphi_a, \varphi_a^*, \partial\varphi_a, \dots]$  stays invariant under this transformation, or equivalently that the Lagrangian  $\mathcal{L}$  only changes by a total derivative. The set of all such symmetry transformations forms a group, which for a continuous group can be parametrized by real parameters  $\varepsilon_i$ . In this case we can expand the transformed field as

$$\varphi'_a(\varepsilon) = \varphi_a + \left. \frac{\partial \varphi'_a(\varepsilon)}{\partial \varepsilon_i} \right|_{\varepsilon=0} \varepsilon_i + \mathcal{O}(\varepsilon^2). \quad (1.5)$$

We call  $\delta_i \varphi_a := \left. \frac{\partial \varphi'_a(\varepsilon)}{\partial \varepsilon_i} \right|_{\varepsilon=0}$  infinitesimal change. The infinitesimal change of the Lagrangian can be written as  $\delta_i \mathcal{L} = \partial_\mu X_i^\mu$ .

Noether's theorem states that any symmetry  $G$  of the theory implies the conservation of a current  $j_i^\mu$  given by

$$j_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta_i \varphi_a - X_i^\mu. \quad (1.6)$$

The current  $j_i^\mu$  is conserved in the sense that it fulfills a continuity equation

$$\partial_\mu j_i^\mu = \partial_t j_i^0 - \nabla \mathbf{j}_i = 0. \quad (1.7)$$

The conserved current additionally implies the existence of a conserved charge,

$$Q_i(t) = \int d^d x j_i^0(x) \quad (1.8)$$

constant in time  $t$ . The proof of this theorem can be found in the literature, for example [6]. It gives a deep insight because symmetries are connected to experimentally found conservation laws and fix the theory to a large extent.

We on the other side just need mathematical relations connected to the charge conservation. When the theory is canonically quantized, which means that we define the conjugate variable  $\Pi$  and equal time commutation relations as

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_a^0} \quad (1.9)$$

$$[\varphi_a(x), \Pi_b(y)] = i\delta(x-y)\delta_{ab} \quad (1.10)$$

the charge can be understood as a generator of the symmetry, i.e

$$i[Q_i, \varphi_a(x)] = \delta_i \varphi_a(x). \quad (1.11)$$

The relation can be directly verified by computing the commutator using (1.8) and (1.7). The last equation will play a role in the proof of Goldstone's theorem.

### 1.2.2. Effective action and groundstates

In this subsection we'll follow the description given by Weinberg [7]. Assume that we are given some scalar field theory. In the canonical picture the fields are operator valued, and act on a physical Hilbert space  $H$ . The Hilbert space in particular contains momentum Eigenstates  $|\lambda_p\rangle$ , which construct a basis of  $H$ , i.e.

$$\mathbb{1} = \sum_{\lambda_p} |\lambda_p\rangle \langle \lambda_p|. \quad (1.12)$$

Among these states there is also the physical vacuum  $|\Omega\rangle$ . It is not only a state of zero momentum, it is also disconnected from the particle states. This means that we cannot take a particle state and reach the vacuum by reducing its momentum.

Spontaneous symmetry breaking happens if several different states minimize the energy. However usually the Hilbert space is hard to access, both theoretically and experimentally. Instead it is convenient to look at vacuum expectation values

$$\langle \Omega | \varphi(x_1) \dots \varphi(x_n) | \Omega \rangle, \quad (1.13)$$

which can be computed without precise knowledge of the Hilbert space. In our case we are particularly interested in the vacuum expectation value of a single field

$$\phi := \langle \Omega | \varphi | \Omega \rangle, \quad (1.14)$$

because it serves as an order parameter in our system, so it plays a similar role as the magnetization in the ferro-magnet example. It usually vanishes but becomes non trivial in the case of a quantum field theory that exhibits spontaneous symmetry breaking.

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The effective action is a functional that describes the energy of a system in terms of the vacuum expectation value  $\phi$ . This interpretation of the effective action is taken from [8]. To see what the effective action is and how it is related to states of minimal energy (groundstates) we start with the transition amplitude from vacuum and vacuum, here in the presence of a source  $J$ , which is given by the path integral

$$Z[J] := \int \mathcal{D}\varphi \exp(iS[\varphi] + i\varphi \cdot J) = \langle \text{Vac, out} | \text{Vac, in} \rangle_J. \quad (1.15)$$

The two vacua are distinguished because the Hilbert spaces at two different times are isomorphic but not identical. Using the path integral we can determine vacuum expectation values by taking functional derivatives:

$$\phi := \frac{\langle \text{Vac, out} | \varphi | \text{Vac, in} \rangle_J}{\langle \text{Vac, out} | \text{Vac, in} \rangle_J} = -\frac{i}{Z} \frac{\delta}{\delta J} Z[J] \quad (1.16)$$

Usually only a gaussian path integral can be computed. This is done in the appendix B.2. Because of this the interaction part is usually expanded and taken out of the integral as functional derivatives. The combinatorics of the derivatives have a graphical representation as Feynman diagrams. Right now the partition function  $Z[J]$  still appears in the denominator, so it would be nice if we could take the logarithm of  $Z$ . In fact, any Feynman diagram decomposes into connected pieces. The generating functional of all connected Feynman diagrams is given by  $W[J]$  and the combinatorics work out such that we can write

$$Z[J] = \exp(iW[J]). \quad (1.17)$$

The functional  $W[J]$  is sometimes called energy functional. Imagine an external source  $J$  lifting the vacuum state to a state of defined energy  $E[J]$ . This should happen smoothly and the new state should be kept for a sufficiently long time  $T$ , such that the system has no dynamics during this time interval. Because of the non-vanishing energy in intermediate states, the two vacua then defer by a phase  $\exp(-iE[J]T)$ . The comparison with the definition of  $W[J]$  gives:

$$W[J] = -iE[J]T. \quad (1.18)$$

This equation gives a connection of energy and source  $J$ . The next step is to make the connection to the vacuum expectation value  $\phi = \langle \Omega | \varphi | \Omega \rangle$ . Interestingly there is a one to one correspondence of sources and vacuum expectation values. For every source  $J$  we can define  $\phi_J = \frac{\delta}{\delta J} W[J]$ , the vacuum expectation value of  $\varphi$  in the presence of a source  $J$ , and for every  $\phi$  there exists a  $J_\phi$  such that  $\phi_{J_\phi} = \phi$ . This allows for a Legendre transform which defines the effective action  $\Gamma$

$$\Gamma[\phi] = -\phi \cdot J + W[J_\phi]. \quad (1.19)$$

The dot denotes the multiplication  $\phi \cdot J := \int \phi J$ . The effective action is a functional in the vacuum expectation value of fields. It serves a similar purpose as the free energy from statistical physics. Instead of understanding our system by probing it with a heat reservoir, we want to understand the fields by bringing them into contact with a source  $J$ . As a candidate of the vacuum state  $|\Omega\rangle$  we want to find the state that minimizes the energy expectation value

$$\min_{|\Omega\rangle} \langle \mathcal{H} \rangle_{|\Omega\rangle} = \min_{|\Omega\rangle} \langle \Omega | \mathcal{H} | \Omega \rangle \quad (1.20)$$

under the constraint that

$$\langle \Omega | \varphi(x) | \Omega \rangle = \phi(x) \quad \text{and} \quad \langle \Omega | \Omega \rangle = 1. \quad (1.21)$$

These constraints should hold during the chosen time interval in that the system doesn't evolve in time apart from a phase. We can rephrase this minimization problem by using Lagrange multipliers:

$$E_{\min} = \min_{|\Omega\rangle} \left( \langle \mathcal{H} \rangle_{|\Omega\rangle} - \lambda \langle \Omega | \Omega \rangle - \int d^4x \mu(x) \phi(x) \right) \quad (1.22)$$

Taking the first derivative in  $\Omega$  yields:

$$\mathcal{H}|\Omega\rangle - \int d^4x \mu(x) \varphi(x) |\Omega\rangle = \lambda |\Omega\rangle \quad (1.23)$$

Now we can compare this eigenvalue equation to

$$\left( \mathcal{H} - \int d^3x J(x)\varphi(x) \right) |E_J\rangle = E[J]|E_J\rangle, \quad (1.24)$$

where the left side of the equation is the Hamiltonian during our time interval  $T$  in the presence of sources.  $|E_J\rangle$  is the normalized eigenstate to the eigenvalue  $E[J]$ . Because the system is smoothly brought into a higher energy state we can assume that  $J(x)\varphi(x)$  is time independent. With  $\int d^3x J(x)\varphi(x) = \int d^4x J(x)\varphi(x)/T$  we obtain the same structure as in (1.23). So under the constraint of a vacuum expectation value  $\phi(x)$  we have to choose

$$\mu(x) = J_\phi(x)/T \quad (1.25)$$

$$\lambda = E[J_\phi] \quad (1.26)$$

$$|\Omega\rangle = |E_{J_\phi}\rangle. \quad (1.27)$$

This is great because now we can rephrase the minimization problem in terms of vacuum expectation values, so we have:

$$\min_{\phi} \langle \mathcal{H} \rangle_{|E_{J_\phi}\rangle} = \min_{\phi} \left( E[J_\phi] \cdot 1 + \int d^4x \frac{J_\phi(x)}{T} \phi(x) \right) \quad (1.28)$$

This formula is directly connected to the effective action. We use that  $\mu(x)\varphi(x) = J(x)/T \cdot \varphi(x)$  is time independent, so we can factor  $T$  out. This gives:

$$\min_{\phi} \langle \mathcal{H} \rangle_{|E_{J_\phi}\rangle} = \min_{\phi} \left( E[J_\phi] + \int d^4x \frac{J_\phi(x)}{T} \phi(x) \right) \quad (1.29)$$

$$= \min_{\phi} \frac{1}{T} \left( -W[J_\phi] + \int d^4x J_\phi(x)\phi(x) \right) = \min_{\phi} -\frac{1}{T} \Gamma[\phi] \quad (1.30)$$

So instead of looking for minima of the energy expectation value in the presence of sources, we can equivalently understand the minima of the effective action. If  $\phi(x) = \phi$  is constant it is possible to write the effective action in terms of the effective potential  $V(\phi)$ :

$$\Gamma[\phi] = \mathcal{V}_3 T V(\phi) \quad (1.31)$$

In particular when sources are absent the vacuum state will relax to a state such that the corresponding vacuum expectation value minimizes  $V(\phi)$  [8]. With the effective potential we have a tool at hands to understand why degenerate vacua appear.

From the discussion before we can easily construct potentials which yield spontaneous symmetry breaking. One example is the famous Mexican hat potential, where a valley of minima appears. More generally all we need is a non vanishing vacuum expectation value  $\phi_0 \neq 0$ , which is a minimizer of the effective potential  $V$  and doesn't respect the full symmetry of the theory. One can show [7] that a symmetry of the action and the path integral measure implies the same symmetry for the effective action. Therefore, the action of the symmetry group  $G$  on  $\phi_0$  gives another minimum of the effective action. In the effective action formalism exist some slight difficulties with the convexity of the effective action, a discussion can be found in [8]. We end up with several degenerate vacuum expectation values which relate to a vacuum state. However we can define fields only as fluctuations above the vacuum expectation value, so the system needs to spontaneously choose one ground state. In theory this is done by explicitly taking a convenient ground state, and reformulating the theory with this reference point. A symmetry breaking is now inevitable. The new theory has still the same symmetry as before, but it is realized in a „non linear“ way.

### 1.2.3. Goldstone's theorem

Now we have everything we need for Goldstone's theorem, again following the proof in [7]. For a theory with symmetry group  $G$  and degenerate vacua, such that any vacuum is only invariant under the action of a subgroup  $H$  of  $G$ , Goldstones theorem predicts  $|G| - |H|$  massless bosonic quasiparticles

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with linear dispersion relation. One example are the transversal modes in the ferromagnet, which are  $|O(3)| - |O(2)| = 2$ . To proof this statement we should remember that any continuous symmetry implies the existence of a conserved current  $J_n^\mu$ . The greek index indicates that this is actually a Lorentz vector. The latin index labels the different kinds of currents, one for each generator of the symmetry group. The induced charge

$$Q_n = \int d^3x J_n^0(x) \quad (1.32)$$

generates the corresponding symmetry transformation, so

$$[Q_n, \varphi_m] = \sum_k t_{n,mk} \varphi_k, \quad (1.33)$$

where  $t_n$  is a generator of the symmetry. This is just a different notation of (1.11). We can then consider the quantity

$$\langle \Omega | [Q_n, \varphi_m] | \Omega \rangle = \int d^3x \langle \Omega | [J_n^0(x), \varphi_m(y)] | \Omega \rangle. \quad (1.34)$$

Our aim will be to treat this quantity similarly to a spectral function. We want to find states that correspond to the Nambu-Goldstone bosons. So let's insert a  $\mathbb{1}$  via intermediate momentum and energy eigenstates as

$$\sum_{|N\rangle} \int d^3x [\langle \Omega | J_n^0(x) | N \rangle \langle N | \varphi_m(y) | \Omega \rangle - \langle \Omega | \varphi_m(y) | N \rangle \langle N | J_n^0(x) | \Omega \rangle]. \quad (1.35)$$

Now we abuse the fact that the four-momentum operator is the generator of spacetime translations, the translational invariance of the vacuum state  $|\Omega\rangle$  and the identities:

$$\langle \Omega | A(x) | N \rangle = \langle \Omega | e^{-ix_\nu P^\nu} A(0) e^{ix_\nu P^\nu} | N \rangle = \langle \Omega | A(0) | N \rangle e^{-ix_\nu p_N^\nu} \quad (1.36)$$

$$\int d^3x e^{-ix_\nu p^\nu} = (2\pi)^3 \delta^{(3)}(\mathbf{p}) e^{-iEt} \quad (1.37)$$

Plugging this in we obtain

$$(1.35) = \sum_{|N\rangle} \left[ \langle \Omega | J_n^0(0) | N \rangle \langle N | \varphi_m(y) | \Omega \rangle \int d^3x e^{-ix_\nu p_N^\nu} - \langle \Omega | \varphi_m(y) | N \rangle \langle N | J_n^0(0) | \Omega \rangle \int d^3x e^{ix_\nu p_N^\nu} \right] \quad (1.38)$$

$$= \sum_{|N\rangle} (2\pi)^3 \delta^{(3)}(\mathbf{p}_N) \left[ \langle \Omega | J_n^0(0) | N \rangle \langle N | \varphi_m(y) | \Omega \rangle e^{-iE_N x^0} - \langle \Omega | \varphi_m(y) | N \rangle \langle N | J_n^0(0) | \Omega \rangle e^{iE_N x^0} \right]. \quad (1.39)$$

The left side on the other hand is:

$$\langle \Omega | [Q_n, \varphi_m] | \Omega \rangle = \sum_k t_{n,mk} \langle \Omega | \varphi_k | \Omega \rangle \quad (1.40)$$

Because we assumed degenerate vacua the vector  $(\langle \Omega | \varphi_k | \Omega \rangle)_{k=1\dots N} \neq 0$ . Here we find the connection to spontaneous symmetry breaking. If, and only if  $t_n$  is the generator of a broken symmetry transformation, so  $g_n \phi_m \neq \phi_m$ , we have:

$$\sum_k t_{n,mk} \langle \Omega | \varphi_k | \Omega \rangle \neq 0. \quad (1.41)$$

Now we can draw the conclusion that Goldstone bosons must exist. Because  $p_n^\nu$  is a Lorentz vector, but the states  $|N\rangle$  carry no momentum due to the delta function, the energy of these states is given

by their mass:  $E_N = m_N$ . The whole equation on the other hand must be independent of  $x^0$ , because  $Q$  is a conserved quantity and thus independent of time. So for  $m_N \neq 0$  we have:

$$\langle \Omega | J_n^0(0) | N \rangle \langle N | \varphi_m(y) | \Omega \rangle = 0 \quad \text{for all } |N\rangle \text{ with } m_N \neq 0 \quad (1.42)$$

For every broken symmetry with generator  $t_n$  the whole expression must be non zero, so there exist states  $|N\rangle$  that fulfill

$$\langle \Omega | J_n^0(0) | N \rangle \neq 0. \quad (1.43)$$

This is true for every conserved current corresponding to a broken symmetry. The states  $|N\rangle$  inherit the quantum numbers of  $\langle \Omega | J_n^0(0)$  and hence must be massless scalars. So for every broken generator  $t_n$  with associated current  $J_n^0$  there exists exactly one massless scalar called Nambu-Goldstone boson.

#### 1.2.4. Dynamics of goldstone bosons

We next want to study the dynamics of goldstone bosons. As massless excitations they will be especially important for the field theory at low energies. In fact, it is a common approach to make the transition to a low energy effective field theory that is purely formulated in terms of Goldstone bosons. This is for example done in the non linear sigma model that approximately describes the dynamics of pions. Pions are very light particles as they are the Goldstone bosons of an approximate broken symmetry in QCD. So it is no surprise that for a long time pions were considered to be fundamental exchange bosons of the strong interaction before sufficiently high energies were reached to hint at the existence of quarks.

Assume we want to make explicit how the theory depends on the massless degrees of freedom. What would the goldstone fields correspond to geometrically? First of all they only appear in the broken phase, so we need to choose one ground state and consider fluctuations above this vacuum expectation value. Next we know that the goldstone bosons are massless. The Eigenvalues of the curvature of the effective action distinguish the massive and massless degrees of freedom, where massless degrees of freedom correspond to directions in which the curvature vanishes. Finally we remember the fact that the Goldstone modes are connected to infinitesimal transformations of the ground state.

This motivates to follow the derivation given by Weinberg [9] to write the fields as a product:

$$\varphi_a(x) = \psi_b(x) \gamma_{ba}(x) \quad (1.44)$$

with  $\psi$  describing heavy modes of the theory and  $\gamma(x)$  being some transformation acting on them. The Goldstone modes will be given by the parameters of this transformation.  $\gamma$  is a  $G$  valued function, which can be seen the following way: We know that ground state is invariant under a group  $H$ . Let's choose generators  $\mathfrak{t}^i$  of  $G$  including the generators  $\mathfrak{h}^i$  of  $H$ . Because the ground state is invariant under  $H$  it also doesn't change infinitesimally under the action of  $H$ , so we have:

$$\mathfrak{h}^i \phi^{(0)} = 0 \quad (1.45)$$

$$\mathfrak{t}^i \phi^{(0)} \neq 0 \quad \text{for } \mathfrak{t}^i \neq \mathfrak{h}^i \quad (1.46)$$

The condition that  $\psi$  should contain only heavy modes can then be written as

$$\psi_a(x) \mathfrak{t}_{ab}^i \phi_b^{(0)} = 0. \quad (1.47)$$

This equation can be interpreted as the condition to be a stationary point of

$$f_{\varphi(x)}(g) = \varphi_a(x) g \phi_b^{(0)}. \quad (1.48)$$

Let's assume that the function becomes stationary at  $g = \gamma^{-1}(x)$ . We can then write group elements in a neighborhood of  $\gamma(x)$  as

$$g_{ab}(\varepsilon) = \gamma_{ac}^{-1}(x) \delta_{cb}(x) = \gamma_{ab}^{-1}(x) + \gamma_{ac}^{-1}(x) \mathfrak{t}_{cb}^i \varepsilon_i + \dots \quad (1.49)$$

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Consequently

$$\begin{aligned} 0 &= \left. \frac{df_{\varphi(x)}(g(\varepsilon))}{d\varepsilon_i} \right|_{\varepsilon=0} = \varphi_a(x) \gamma_{ac}^{-1}(x) \mathbf{t}_{cb}^i \phi_b^{(0)} \\ &= \psi(x) \mathbf{t}_{cb}^i \phi_b^{(0)} \end{aligned} \quad (1.50)$$

which yields

$$\varphi_a(x) = \psi_b(x) \gamma_{ba}(x). \quad (1.51)$$

As conjectured  $\gamma$  is  $G$  valued. We can go one step further by recognizing that

$$f_{\varphi(x)}(g) = f_{\varphi(x)}(gh) \quad (1.52)$$

for  $h \in H$ . Because of this  $\gamma$  is not uniquely defined, but only up to the action of  $H$ . At this point it is convenient to introduce the coset space  $G/H$  which consists out of equivalence classes  $g \cdot H$ . This coset space can be equipped with a manifold structure. Thus we can restrict  $\gamma$  to vary smoothly on  $G/H$ . We can now write

$$\varphi_a(x) = \psi_b(x) \gamma_{ba}(\xi_i(x)), \quad (1.53)$$

where  $\xi_i$  are the local coordinates of a parametrization of  $G/H$ . As we included all heavy modes into  $\psi(x)$  the fields  $\xi_i$  must be directly related to the goldstone fields. So far it stays unclear how the action of  $G$  should be understood now, as we reduce the group  $G$  to  $G/H$ . This is explained by so called non linear realizations of  $G$  on  $G/H$ . The transformation can then depend on the space time point  $x$  leading to the appearance of covariant derivatives.

### 1.2.5. Non linear realizations

The following treatment was for example given by Coleman 1969 [10]. We take a group element  $g \in G$  and write it as a product of  $h \in H$  and a parametrization of  $G/H$   $U(\xi_a)$ :

$$g = U(\xi_a)h \quad (1.54)$$

If we now let another group element  $g'$  act on  $U(\xi_a)$  we have because of the closure of the groups a decomposition:

$$g'' = g'U(\xi_a) = U(\xi'_a(\xi_a, g'))h'(\xi_a, g') \quad (1.55)$$

Assume now that the action of  $H$  is realized linearly on the field space, so we have

$$h : \psi \rightarrow D(h)\psi, \quad (1.56)$$

then this induces a non linear realization of  $G$  as

$$g : (\xi_a, \psi) \rightarrow (\xi'_a(\xi_a, g), D(h'(\xi_a, g))\psi). \quad (1.57)$$

In general this can be a very complicated nonlinear function. Additionally the transformation  $\xi_a \rightarrow \xi'_a$  only depends on  $\xi_a$  and  $g$ . Thus the action of  $g$  can also be defined on the coset space  $G/H$  alone. One way to obtain effective Lagrangians from this is by finding non linear expression of the action of  $g$ , the corresponding covariant derivatives and the simplest invariants under the action of  $g$ . This is usually done using the Maurer Cartan form and a description can be found in [11]. We will use a different more direct derivation later on.

## 1.3. The Gross-Pitaevskii Lagrangian

In the context of quantum field theory, the behavior of a Bose Einstein condensate without external potential is usually described by the Gross-Pitaevskii equation or non linear Schrödinger equation:

$$i \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = \left( -\frac{\nabla^2}{2m} + g|\psi(\mathbf{x}, t)|^2 \right) \psi(\mathbf{x}, t) \quad (1.58)$$



There are several approaches to this equation. We want to understand this equation as the classical field equation corresponding to the nonrelativistic limit of  $\varphi^4$  theory using the derivation in [12]. We want to describe a gas of bosons, each of them being quantum mechanically described by a complex wave function. The natural field to model the bosons and their interactions is a complex scalar field. We know from quantum field theory that there are only two different options for such a field theory,  $\varphi^3$  and  $\varphi^4$ . The latter describes an interaction that is proportional to the density squared. This suits to our idea of collisions of classical particles, so it is the convenient choice to model our gas. This approach is however only valid for dilute gases. We now want to study the non relativistic limit of  $\varphi^4$  theory

$$\mathcal{L} = \left( \dot{\varphi}^\dagger \dot{\varphi} - \nabla \varphi^\dagger \nabla \varphi - m^2 \varphi^\dagger \varphi \right) - \frac{\lambda}{12} (\varphi^\dagger \varphi)^2 \quad (1.59)$$

at  $T = 0$ , so we don't need to bother about boundary conditions. A more general discussion is given by Evans [13]. The conjugate momenta are

$$\begin{aligned} \pi &= \frac{\partial}{\partial \dot{\varphi}} \mathcal{L} = \dot{\varphi}^\dagger \\ \pi^\dagger &= \dot{\varphi}. \end{aligned} \quad (1.60)$$

In the non relativistic limit we expect the following approximations: The dispersion relation is  $E = \sqrt{\mathbf{p}^2 + m^2} \simeq m + \frac{\mathbf{p}^2}{2m}$ . Antiparticles and their dynamics are absent (for the fundamental fields), so the relativistic chemical potential will be of order of the mass  $\mu \simeq m$ . The energy associated to the change of particle number by one is mostly determined by it's mass and it is reasonable to define the non relativistic chemical potential  $\mu_{nr} = \mu - m$ . Furthermore  $\varphi^4$  theory has a  $U(1)$  symmetry. Using Noether's theorem (1.6) we obtain the corresponding charge

$$Q = i \int d^3x \left[ \pi^\dagger \varphi^\dagger - \pi \varphi \right]. \quad (1.61)$$

This charge ensures charge conservation, if we associate particles with positive charge and antiparticles with negative charge. We assume antiparticles to be absent in the non-relativistic theory. In this case the charge conservation is identical with particle number conservation which is reasonable for a gas at low temperatures. We employ this conservation law in the non relativistic theory via the chemical potential as a Lagrange multiplier. So

$$S = \int dt \int d^d x \mathcal{L} = \int dt \int d^d x \left( \dot{\varphi}^\dagger \pi^\dagger + \dot{\varphi} \pi \right) - [H - \mu Q] \quad (1.62)$$

with

$$H = \int d^d x \left( \pi^\dagger \pi - \nabla \varphi^\dagger \nabla \varphi - m^2 \varphi^\dagger \varphi - \frac{\lambda}{12} (\varphi^\dagger \varphi)^2 \right) \quad (1.63)$$

The quadratic part of this Lagrangian is considered as the free part, while the quartic term is called the interaction term. The free Lagrangian in  $\varphi$  and  $\pi$  can be diagonalized in momentum space, introducing

$$E_p = \sqrt{\mathbf{p}^2 + m^2} \quad (1.64)$$

$$a(\mathbf{p}, t) = \sqrt{\frac{E_p}{2}} \varphi(\mathbf{p}, t) + \frac{i}{\sqrt{2E_p}} \pi^\dagger(\mathbf{p}, t) \quad (1.65)$$

$$b(\mathbf{p}, t) = \sqrt{\frac{E_p}{2}} \varphi^\dagger(\mathbf{p}, t) + \frac{i}{\sqrt{2E_p}} \pi(\mathbf{p}, t) \quad (1.66)$$

and their complex conjugates. This splits the Lagrangian into two contributions, where one can be considered as corresponding to particles, while the other is interpreted as antiparticles:

$$S = \int dt \int d^d p i \left[ a^\dagger(\mathbf{p}, t) (i\partial_t - E_p + \mu) a(\mathbf{p}, t) + b^\dagger(\mathbf{p}, t) (i\partial_t - E_p - \mu) b(\mathbf{p}, t) \right] \quad (1.67)$$

## 1. Introduction

Due to our definition of the chemical potential the first term corresponds to particles. To treat the interactions we express  $\varphi$  as

$$\varphi = \frac{1}{\sqrt{2E_p}} (a + b^\dagger) \quad (1.68)$$

$$\varphi^\dagger = \frac{1}{\sqrt{2E_p}} (a^\dagger + b) \quad (1.69)$$

and plug it into the interaction term  $\frac{\lambda}{12}(\varphi^\dagger\varphi)^2$ . This gives the expression

$$\frac{\lambda}{12}(\varphi^\dagger\varphi)^2 = \frac{\lambda}{48E_p}(a^\dagger a + a^\dagger b + b^\dagger a + b^\dagger b). \quad (1.70)$$

So particles and antiparticles don't decouple. Nevertheless antiparticles are really heavy in the sense that employing  $E_p = m + \mathbf{p}^2/2m$  the particle term has mass  $\mu_{nr}$  that is assumed to be small compared to  $m$ , while the antiparticle term has mass  $\mu + m \simeq 2m$ . For energies that are much smaller than the mass  $m$ , contributions of the antiparticles are suppressed and we can only keep particle contributions. It can be shown that this describes the behavior of  $\varphi^4$  theory at low energies very well [12]. With this we finally arrive at the Lagrangian

$$\mathcal{L}_{GP} = \tilde{\varphi}^\dagger (i\partial_t + \frac{\nabla^2}{2m} + \mu_{nr})\tilde{\varphi} - \frac{g}{2}(\tilde{\varphi}^\dagger\tilde{\varphi})^2 \quad \text{with} \quad g = \frac{\lambda}{24m^2} \quad (1.71)$$

with  $\tilde{\varphi}(\mathbf{x}, t)$  being the Fourier-transform of  $a(\mathbf{p}, t)$ . This equation describes a nonrelativistic gas of bosons. So we would expect to also understand Bose Einstein condensation from this equation. We already have a chemical potential term in the equation, which ensures the conservation of total particle number  $N = \int d^d x \tilde{\varphi}^\dagger \tilde{\varphi}$ . We will now see that the chemical potential is connected to condensation just as in statistical physics. In the following we will rename the fields back to  $\varphi$ , which should now be understood as the fundamental fields appearing in the Gross-Pitaevskii Lagrangian. The above approach is not the only one to the Gross-Pitaevskii model. Another very convenient derivation, coming from quantum mechanics, lies in finding a Lagrangian that yields the Schrödinger-equation as equations of motion with the same conserved probability-current. See for example [14]. This leads to a slightly more general formulation.

### 1.3.1. Bose-Einstein Condensation

In statistical physics a relation between chemical potential and particle number can be derived from the grand canonical partition function. In this context the chemical potential becomes a function of temperature,  $\mu = \mu(T)$  and raises from  $-\infty$  to zero as the temperature goes down. At the critical temperature the chemical potential theoretically reaches the value zero. The chemical potential may not become positive, so a phase transition is reached where particles condense into the ground state of the system. In Quantum Field theory the particle number operator should be replaced by its local version, the density operator  $\rho = \varphi^\dagger\varphi$ . Usually the vacuum expectation value of the density operator vanishes

$$\langle \Omega | \rho | \Omega \rangle = 0. \quad (1.72)$$

If it doesn't vanish we are in the broken phase and the density of particles in the vacuum state is positive, which we can understand as Bose-Einstein condensation. Let's now consider the vacuum of the Gross-Pitaevskii equation. We have seen in the first section that it is sufficient to analyze the minima of the effective potential. When reintroducing  $\hbar$  the effective action can be written as

$$\Gamma[\phi] = S[\phi] + \hbar K[\phi], \quad (1.73)$$

where  $K[\phi]$  is some function appearing from quantum effects. It corresponds to loop-diagrams and is waited with  $\hbar$ . Consequently the effective potential behaves basically just as the potential (the free term doesn't shift the minima) from our Lagrangian, except for the case of  $\mu = 0$  which needs to

be treated seperatedly. For negative  $\mu < 0$  the vacuum is trivial and the order parameter  $\langle \rho \rangle = 0$  vanishes. For positive  $\mu$  however we observe spontaneous symmetry breaking of the  $U(1)$  symmetry: The minimum of the potential lies at  $\phi_0$  which satisfies

$$(g\phi_0^\dagger\phi_0 - \mu)\phi_0 = 0. \quad (1.74)$$

The number of vacua is degenerate and we need to choose one randomly. If we take  $\phi_0 = \sqrt{\frac{\mu}{g}}$  the particle density becomes positive in the vacuum which is equivalent to the ground state of the bose gas. The Goldstone boson is given by the Bogoliubov mode. Before we said that positive  $\mu$  in statistical physics are not allowed. We would actually expect this to still hold, as a positive chemical potential was associated with antiparticles before. This poses no problem because also for the critical value  $\mu = 0$  spontaneous symmetry breaking is observed. This is done by explicitly computing the effective potential, for example to one loop order. This procedure yields the so called Coleman-Weinberg potential, where the number of minima is in fact degenerate. We can conclude: Bose Einstein condensation can be interpreted as spontaneous symmetry breaking in a non relativistic bose gas. However we cannot connect the chemical potential to the critical temperature, which is also not in the scope of interest here.



## 2. Low energy effective theory

My work consists mainly in deriving the effective theory of the Bose Einstein condensate in terms of Goldstone bosons. For this we will analyze the symmetries of theory and ground state and parametrize the corresponding coset space. Afterwards the in the overview mentioned Bogoliubov mode and free particle modes can be identified. After integrating out density fluctuations using the path integral we'll arrive at an effective theory, which allows for conclusions regarding the interactions between the quasi particle fields.

### 2.1. Symmetry breaking in the $N$ component Gross-Pitaevskii model

#### 2.1.1. Symmetries of the theory

The introduction was given in one complex field for simplicity and readability. It is easily possible to generalize all topics to a theory of  $N$  complex fields by adding an index and assuming Einstein's sum convention. This also leads to the  $N$  component Gross-Pitaevskii model given by the Lagrangian:

$$\mathcal{L} = \frac{i}{2} (\varphi_a^* \dot{\varphi}_a - \varphi_a \dot{\varphi}_a^*) - \frac{1}{2m} (\nabla \varphi_a^*) (\nabla \varphi_a) - \frac{g}{2} (\varphi_a^* \varphi_a)^2 + \mu \varphi_a^* \varphi_a. \quad (2.1)$$

The equation is a slight variation of (1.71), where a total time derivative  $\partial_t(\varphi^\dagger \varphi)$  was added and one  $\nabla$  was shifted by partial integration. It exhibits spontaneous symmetry breaking for  $\mu \geq 0$ . I mentioned before that the Gross-Pitaevskii equation has a  $U(1)$  symmetry. This applies to every component now. Additionally there is no preferred direction in field space which is usually described by an  $O(N)$  symmetry. However mathematically the symmetry is even bigger, because even real and imaginary parts of the components can be mixed. To see this we should remember the definition of the complex dot product of two  $N$  dimensional complex vectors  $v, w$ .

$$\langle v, w \rangle = v_a^\dagger w_a \quad (2.2)$$

This are exactly the scalars that appear in our Lagrangian. The group that keeps the complex dot product invariant is the unitary group:

$$\langle Uv, Uw \rangle = v^\dagger U^\dagger U w = \langle v, w \rangle \quad (2.3)$$

where  $U$  denotes the unitary transformation. The  $O(N)$  symmetry of the Lagrangian is only a subgroup of the bigger symmetry  $U(N)$ . Because the Lagrangian consists only of dot products of the type (2.2) it is also the biggest symmetry possible. Next we need the symmetry of the ground state. For this we minimize the potential

$$V(\phi^\dagger, \phi) = \frac{g}{2} (\phi_a^\dagger \phi_a)^2 - \mu \phi_a^\dagger \phi_a \quad (2.4)$$

which behaves for  $\mu \neq 0$  like the effective potential. The first derivative of the potential is:

$$\frac{\partial V}{\partial \phi_b^\dagger} = g(\phi_a^\dagger \phi_a) \phi_b - \mu \phi_b \quad (2.5)$$

So the ground state is non trivial and degenerate, because all

$$\phi^{(0)} = (\phi_a^{(0)})_{a=1\dots N} \quad (2.6)$$

with

$$(g\phi_a^{(0)\dagger} \phi_a^{(0)} - \mu) \phi_b^{(0)} = 0 \quad (2.7)$$

## 2. Low energy effective theory

for all  $b = 1 \dots N$  fulfill the necessity condition to be a minimum. This means that we'll observe spontaneous symmetry breaking. The vacuum expectation values don't vanish and describe the ground state's density  $\rho_0 := \phi_a^{(0)\dagger} \phi_a^{(0)}$ . The next steps are easier with an explicit choice of the ground state, but do not depend on the specific choice made. Let's consider for example

$$\phi^{(0)} = \left( \sqrt{\frac{\mu}{g}}, 0, \dots, 0 \right)^T. \quad (2.8)$$

The remaining symmetry can easily be read of. We see that in the first component we require a trivial transformation behavior. So any symmetry transformation of the ground state must have the shape:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} \quad (2.9)$$

with  $W$  being an  $N - 1$  by  $N - 1$  square matrix. We furthermore require  $U$  being a unitary transformation, which directly implies:

$$U^\dagger U = \mathbb{1}_N \implies W^\dagger W = \mathbb{1}_{N-1} \quad (2.10)$$

This means nothing else but that the symmetry (sub)group of the ground state is  $U(N - 1)$ . As a consequence the symmetry breaking of the  $N$  component Gross-Pitaevskii model is

$$U(N) \rightarrow U(N - 1). \quad (2.11)$$

The number of Goldstone modes is given by the dimension of the coset space. The dimension of the unitary group as a real manifold is  $\dim U(m) = m^2$ , so Goldstones theorem applied to this symmetry breaking behavior predicts

$$N^2 - (N - 1)^2 = 2N - 1 \quad (2.12)$$

Nambu-Goldstone bosons. Nevertheless we saw in the introduction that it is sufficient to consider only  $N$  of them in the effective theory. In the next sections we want to study the dynamics of the Goldstone bosons to see if there is redundance in the degrees of freedom.

## 2.2. Parametrizing the coset space

As we have seen in section 1.2.4 we need a parametrization of the coset manifold in some neighborhood of the ground state. This could always be done via exponential maps when knowing the generators of  $G$  that don't generate  $H$ . We want to go a different route here. We already know from the previous work [1] presented in 1.1 that we expect one Bogoliubov mode and  $N - 1$  modes with quadratic dispersion relation. This could result if we single out the Bogoliubov mode and observe the  $2N - 2$  other modes to describe pairwise only one degree of freedom.

### 2.2.1. Hopf fibration of $S^{2N-1}$

We first state the mathematical fact that

$$U(N)/U(N - 1) = S^{2N-1}. \quad (2.13)$$

We remember that the Bogoliubov mode describes the global phase. A phase can always be seen as a point on a circle  $S^1$ . We would like to "divide" in some sense  $S^{2N-1}$  by  $S^1$ . This can be done by a Hopf like fibration; we view the  $S^{2N-1}$  sphere as a  $S^1$  bundle over the complex projective space  $\mathbb{C}\mathbb{P}^{N-1}$ . To simplify this mathematical jargon we have to understand what the complex projective space is and see how it is connected to spheres.

The elements of the projective space  $\mathbb{C}\mathbb{P}^{N-1}$  are the complex planes through the origin. For a complex plane  $\text{span}_{\mathbb{C}}(w = (Z_1, \dots, Z_N)^T) \subseteq \mathbb{R}^{2N}$  all points are associated with each other and represented by a *representative* geometrically. In principle the choice of the representative is completely arbitrary, but usually it is possible to give a construction which relates this abstract space to geometric shapes.

To do this in our case we normalize  $w$ :  $w' = \frac{w}{\|w\|}$ .  $w'$  now lies on the sphere  $S^{2N-1}$ . The intersection of the complex plane  $\text{span}_{\mathbb{C}}(w)$  and  $S^{2N-1}$  defines a circle of points  $S^1$ . Any point on that circle can now be chosen as the representative. Here we just take all points of the circle to be associated with one point of the complex projective space. This defines a projection:

$$\pi : S^{2N-1} \rightarrow S^{2N-1}/S^1 \cong \mathbb{C}\mathbb{P}^{N-1}, \quad w' \mapsto w'/S^1. \quad (2.14)$$

This projection cannot be inverted globally but only locally.  $S^{2N-1}$  can be understood as this set of representatives called projective space with a circle  $S^1$  attached to each point. Mathematically this, together with some requirements on the smoothness of the attachment, defines a fibre bundle on the  $\mathbb{C}\mathbb{P}^{N-1}$  with fibre  $S^1$ . Compared to a product  $\mathbb{C}\mathbb{P}^{N-1} \times S^1$  a fibre bundle has to be seen as a collection of circles  $(S^1_z)_{z \in \mathbb{C}\mathbb{P}^{N-1}}$  where copies are distinguished by the base point  $z$ . This type of decomposition of a sphere is also called Hopf fibration.

### 2.2.2. Construction of the parametrization

Although a fibre bundle is not a product of topological spaces the circles must be glued together in such a way that the fibre bundle locally looks like a product, locally in the same sense as a manifold locally looks like a euclidean space. This means that around the ground state we are allowed to write the point of the sphere as a product:

$$x = e^{i\theta} p \quad \text{with } x \in S^{2N-1}, \quad e^{i\theta} \in S^1, \quad p \in \mathbb{C}\mathbb{P}^{N-1} \quad (2.15)$$

We parametrize the complex projective space using homogenous coordinates. We consider the subspace  $\{p = 1e_1 + \lambda_i e_i \mid \lambda_i \in \mathbb{C}\}$  with standard basis  $e_i$ , which is normal to the direction of the groundstate. This space will now be intersected by all complex planes  $(Z_1, \dots, Z_N)$  with  $Z_1 \neq 0$  at some  $(1, z)$ ,  $z \in \mathbb{C}^{N-1}$ . The first component may not be trivial, but a global parametrization is impossible anyway. By normalizing the point  $(1, z)$  we find a representative of the complex plane spanned by this point

$$p(z) = \frac{1}{\sqrt{1 + z^\dagger z}} \begin{pmatrix} 1 \\ z \end{pmatrix} \quad \text{with } z \in \mathbb{C}^{N-1} \quad (2.16)$$

which lies on the unit sphere. The fixed first component on the other hand fixes the phase because any phase rotation of  $(1, z)$  is given by the scalar multiplication with a complex number  $e^{i\varphi}$  which involves a phase rotation of the first component. In conclusion  $p(z)$  defines a parametrization of the complex projective space. Back to physics: Now we are able to express our fields in terms of Goldstone modes. First we consider density fluctuations above the ground state:

$$\rho = \rho_0 + \delta\rho \quad \text{with } \rho_0 = \frac{\mu}{g} \quad (2.17)$$

Then we follow the idea of subsection 1.2.4 and let our coset parametrization act on the ground state with fluctuations:

$$(\varphi_a)_{a=1\dots N} = \sqrt{\rho} e^{i\theta} \frac{1}{\sqrt{1 + z^\dagger z}} \begin{pmatrix} 1 \\ z \end{pmatrix} \quad (2.18)$$

When we plug in  $\theta = 0, z = 0, \delta\rho = 0$  we are in the ground state. The parameters describe fluctuations above this ground state but are constraint to a higher dimensional sphere. This property will yield some complications later. This specific choice to parametrize the fields was already given by Watanabe and Murayama in [11]. It could also be constructed without knowledge of fibre bundles; however the above geometric picture shows that the Bogoliubov mode lives in the fibre  $S^1$ , while the free particle modes live in the symplectic manifold  $\mathbb{C}\mathbb{P}^{N-1}$  which is an indicator that they form conjugated pairs of variables. The symplectic structure is given by the Fubini Study metric which arises naturally later on from derivatives of the denominator.

## 2. Low energy effective theory

### 2.2.3. Independence of groundstate

This construction doesn't depend on the especially simple choice of the ground state. Assume the more general case

$$\phi'_0 = \left( \sqrt{\rho_1^{(0)}}, \dots, \sqrt{\rho_N^{(0)}} \right)^T \quad (2.19)$$

and define  $n_0 := \phi'_0/\rho_0$ . In general we should include a constant phase factor in every component, which is neglected as a non-measurable quantity. In this case we can choose  $N-1$  vectors  $n_1 \dots n_{N-1}$  such that  $n_0, n_1, \dots, n_{N-1}$  are orthonormal vectors with respect to the real euclidean dot product. This is possible because we assumed a real ground state, such that the standard basis of  $\mathbb{R}^N$  is also a  $\mathbb{C}$ -basis of  $\mathbb{C}^N$ . We now consider an  $N$  dimensional complex vector  $Z \in \mathbb{C}^N$  with  $Z \cdot n_0 \neq 0$  which can be expressed as:

$$\begin{aligned} Z &= Z_1 n_0 + \dots + Z_N n_{N-1} = Z_1 (n_0 + Z_2/Z_1 n_1 + \dots + Z_N/Z_1 n_{N-1}) \\ &= Z_1 (n_0 + \xi_1 n_1 + \dots + \xi_{N-1} n_{N-1}). \end{aligned} \quad (2.20)$$

A complex plane spanned by  $Z$  therefore intersects the hyperplane

$$\{p = n_0 + \lambda_1 n_1 + \dots + \lambda_{N-1} n_{N-1} \mid \lambda_i \in \mathbb{C}\} \quad (2.21)$$

at  $p = (1, \xi)$  with  $\xi = (\xi_1, \dots, \xi_{N-1})$ . The norm of  $p$  is given by

$$\|p\|^2 = 1 + \sum_i |\xi_i|^2 = 1 + \xi^\dagger \xi \quad (2.22)$$

where Pythagoras' theorem was used and the fact that all basis vectors are orthonormal. The rest of the construction stays the same. We obtain a parametrization of the fields as

$$(\varphi_a)_{a=1\dots N} = \sqrt{\rho} e^{i\theta} \frac{1}{\sqrt{1 + \xi^\dagger \xi}} \begin{pmatrix} 1 \\ \xi \end{pmatrix}_{n_0, \dots, n_{N-1}} \quad (2.23)$$

The new basis is related to the standard basis by a simple orthogonal transformation (in particular unitary), so the basis exchange matrix will be unitary. It will drop out when plugging in the parametrization. All upcoming results don't depend on the choice of the ground state, but only on  $\rho_0$ . This also implies that in this model initially empty components are not negligible but densities can be shifted to other components which corresponds to higher values of  $z$ .

## 2.3. Low energy effective theory

The next aim is to express the entire model in terms of Goldstone bosons and verify that they are the same objects as the modes introduced in section 1.1. For this purpose we plug into the Lagrangian our newly found parametrization. Throughout this section I'll use the fact that the Lagrangian for itself is no physical observable. We always need to think of the Lagrangian as the integrand of the action. Because of this total derivative terms are surface terms and can be disregarded as long as we do not take topological effects into consideration. I will also frequently use partial integration without integral sign although they only hold as an integrand. The time derivative term can be obtained by direct computation:

$$\frac{i}{2} (\varphi_a^\dagger (\partial_t \varphi_a) - \partial_t (\varphi_a^\dagger) \varphi_a) = -\rho \dot{\theta} + \frac{i}{2} \frac{\rho}{1 + z^\dagger z} (z^\dagger \partial_t z - \partial_t (z^\dagger) z) \quad (2.24)$$

The spatial derivative is more complicated due to less cancelations. We first take the simple spatial derivative:

$$\begin{aligned} \partial_i \varphi_a &= \frac{\partial_i \sqrt{\rho}}{\sqrt{1 + z^\dagger z}} e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a - \frac{1}{2} \frac{\sqrt{\rho}}{\sqrt{1 + z^\dagger z}^3} [(\partial_i z^\dagger) z + z^\dagger (\partial_i z)] e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a \\ &\quad + \sqrt{\frac{\rho}{1 + z^\dagger z}} (i \partial_i \theta) e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a + \sqrt{\frac{\rho}{1 + z^\dagger z}} e^{i\theta} \begin{pmatrix} 0 \\ \partial_i z \end{pmatrix}_a \end{aligned} \quad (2.25)$$



This result is multiplied with its complex conjugate and summarized. Details of this derivation can be found in Appendix C.2.

$$\begin{aligned}
 (\partial_i \varphi_a^\dagger)(\partial_i \varphi_a) &= (\partial_i \sqrt{\rho})^2 + \rho \left[ \left( \partial_i \theta + \frac{i}{2} \frac{(\partial_i z^\dagger) z - z^\dagger (\partial_i z)}{(1 + z^\dagger z)^2} \right)^2 \right. \\
 &\quad \left. + \frac{(\partial_i z^\dagger)(\partial_i z)}{1 + z^\dagger z} - \frac{(\partial_i z^\dagger) z \cdot z^\dagger (\partial_i z)}{(1 + z^\dagger z)^2} \right]. \tag{2.26}
 \end{aligned}$$

Here the notation  $(\partial_i z^\dagger)z = \sum_a (\partial_i z_a^\dagger)z_a$  is used and it is summed over the index  $i$ . Using additionally that  $\varphi_a^\dagger \varphi_a = \rho$  we are now in the position to construct the Lagrangian parametrized by the  $\theta$  and  $z$  fields.

$$\begin{aligned}
 \mathcal{L} &= -\rho \dot{\theta} + \frac{i}{2} \frac{\rho}{1 + z^\dagger z} (z^\dagger \partial_t z - \partial_t (z^\dagger) z) \\
 &\quad - \frac{1}{2m} \left\{ (\nabla \sqrt{\rho})^2 + \rho \left[ \left( \nabla \theta + \frac{i}{2} \frac{(\nabla z^\dagger) z - z^\dagger (\nabla z)}{(1 + z^\dagger z)^2} \right)^2 \right. \right. \\
 &\quad \left. \left. + \frac{(\nabla z^\dagger)(\nabla z)}{1 + z^\dagger z} - \frac{(\nabla z^\dagger) z \cdot z^\dagger (\nabla z)}{(1 + z^\dagger z)^2} \right] \right\} + \frac{g}{2} \rho^2 - \mu \rho \tag{2.27}
 \end{aligned}$$

We need to be careful with the term  $(\partial_i \sqrt{\rho})^2$  as it should be treated perturbatively. For this consider that  $\rho$  itself shouldn't be understood as a field, because the vacuum state is realized for  $\rho = \rho_0 \neq 0$ . We should rather split  $\rho$  into the ground state density and fluctuations above it:

$$\rho = \rho_0 + \delta \rho \tag{2.28}$$

The term can then be expanded under the assumption  $\delta \rho / \rho_0 \ll 1$ .

$$(\partial_i \sqrt{\rho})^2 = \left( \frac{1}{2\sqrt{\rho_0}} \partial_i \delta \rho \right)^2 + \mathcal{O}(\delta \rho^3) \approx \frac{1}{4\rho_0} (\partial_i \delta \rho)^2. \tag{2.29}$$

The assumption is discussed later. Additionally using this decomposition the last terms can be simplified:

$$\frac{g}{2} \rho^2 - \mu \rho = \frac{g}{2} \rho_0^2 - \mu \rho_0 + (g\rho_0 - \mu) \delta \rho + \frac{g}{2} \delta \rho^2 \tag{2.30}$$

The first term is constant and can be disregarded in the Lagrangian. It doesn't give any information about the dynamics of the fields, neither classically nor in quantum physics. The second term vanishes because of the definition of  $\rho_0$ . We thus are left with only one relevant term for the Lagrangian, i.e.  $\frac{g}{2} \delta \rho$ . Similarly usually total derivative terms don't contribute to the Lagrangian. However we have to remember that for example  $\theta$  is as a phase  $S^1$  valued. So we can't say that  $\theta$  vanishes at infinity such that boundary terms can be neglected. We assume that these boundary effects play no role for us and proceed without them.

### 2.3.1. Dispersion relations

To verify that our parametrization is reasonable it is convenient to have a quick look at the dispersion relations of the fields. This is done by determining linearized equations of motion from terms that are of quadratic order in fields. Hence for the linearized equations of motion we should use

$$\frac{1}{1 + z^\dagger z} = 1 + \mathcal{O}(\|z\|^2) \tag{2.31}$$

and make an approximation to quadratic order in fields for all terms. This yields

$$\mathcal{L}^{(2)} = -\delta \rho \dot{\theta} + \frac{i}{2} \rho_0 (z^\dagger \dot{z} - \dot{z}^\dagger z) - \frac{1}{2m} \left[ + \frac{1}{4\rho_0} (\nabla \delta \rho)^2 + \rho_0 (\nabla \theta)^2 + \rho_0 \nabla z^\dagger \nabla z \right] + \frac{g}{2} \delta \rho^2. \tag{2.32}$$

## 2. Low energy effective theory

The Euler Lagrange equations for a variable  $\psi$  are

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial \partial_{x_i} \psi} = \frac{\partial \mathcal{L}}{\partial \psi}. \quad (2.33)$$

From this we obtain the equations

$$\frac{i}{2} \frac{d}{dt} (\rho_0 z^\dagger) - \frac{\rho_0}{2m} \nabla^2 z^\dagger = -\frac{i}{2} \rho_0 \dot{z}^\dagger \quad (2.34)$$

$$-\frac{d}{dt} \delta \rho + 2\rho_0 \frac{\nabla^2 \theta}{2m} = 0 \quad (2.35)$$

$$-\frac{2}{8m\rho_0} \nabla^2 \delta \rho = -\dot{\theta} + g\delta \rho. \quad (2.36)$$

Combining (2.35) and (2.36) we find:

$$\begin{aligned} \ddot{\theta} &= -\frac{\nabla^2}{4m\rho_0 g} \dot{\delta \rho} + g\delta \dot{\rho} = \left( g - \frac{\nabla^2}{4m\rho_0} \right) \left( 2\frac{\rho_0}{2m} \nabla^2 \theta \right) \\ &= \frac{\nabla^2}{2m} \left( -\frac{\nabla^2}{2m} + 2g\rho_0 \right) \theta \end{aligned} \quad (2.37)$$

For the  $z^\dagger$  mode we directly have from (2.34):

$$i\dot{z}^\dagger = \frac{1}{2m} \nabla^2 z^\dagger. \quad (2.38)$$

The equivalent equation can be obtained for  $z$ . One obtains back the linearized equations of motion we expected, with one mode having a Bogoliubov dispersion, while the  $2N - 2$  other modes follow a free particle dispersion relation. The geometrical construction leads to the same massless degrees of freedom as the more physical construction with a density phase representation. From now on we will refer to the  $z$  modes as *free particle* modes because of their dispersion relation.

### 2.3.2. Gaussian integration of the density fluctuations

The longitudinal density fluctuation  $\delta \rho$  is a heavy mode in the system because it corresponds to a positive eigen mode of the potential's curvature. The gapless degrees of freedom dominate the theory at our scales. Because of this density fluctuations can be integrated out and an effective theory in massless modes should give reasonable results. If the classical action is called  $S$  this corresponds to

$$\mathcal{C} e^{iS_{\text{eff}}} = \int \mathcal{D}\delta \rho e^{iS(\delta \rho, \theta, z, z^\dagger)} \quad (2.39)$$

with some irrelevant constant  $\mathcal{C}$ . (A discussion of this equation is given in C.1.) The approximation (2.29) turns out to be especially valuable to make the transition to an effective theory. From a semiclassical estimate it can be obtained that  $\frac{\delta \rho}{\rho_0} \sim \frac{\|\mathbf{k}\|}{k_\Xi}$  [15]. We investigate a model at momenta far below the healing length  $k_\Xi$ , so density fluctuations should be strongly suppressed. This justifies to consider the term  $(\partial_i \sqrt{\rho})^2$  perturbatively. The last point is crucial since generally the path integral can only be computed if the action is quadratic in the integration variable. In this case the classical Gaussian integral can be generalized, which is shown in B.2. To allow a more transparent integration we need to make another partial integration in the action.

$$\frac{1}{4\rho_0} (\partial_i \delta \rho)^2 + \mathcal{O}(\delta \rho^3) \cong -\frac{1}{4\rho_0} (\partial_i^2 \delta \rho) \delta \rho + \mathcal{O}(\delta \rho^3) \quad (2.40)$$

The correspondence holds under the integral sign. Using our previous approximation in density fluctuations we split the Lagrangian up:

$$\mathcal{L} = \mathcal{L}^{(G)} + \mathcal{L}^{(n.G)} \quad (2.41)$$

with

$$\mathcal{L}^{(G)} = \rho_0 J(x) + \delta\rho J(x) - \frac{1}{2}\delta\rho\hat{g}\delta\rho \quad (2.42)$$

$$J(x) = -\dot{\theta} + \frac{i}{2} \frac{z^\dagger \dot{z} - \dot{z}^\dagger z}{1 + z^\dagger z} - \frac{1}{2m} \left[ \left( \nabla\theta + \frac{i}{2} \frac{z^\dagger \nabla z - \nabla z^\dagger z}{1 + z^\dagger z} \right)^2 + \frac{(\nabla z^\dagger)(\nabla z)}{1 + z^\dagger z} - \frac{(\nabla z^\dagger)z \cdot z^\dagger(\nabla z)}{(1 + z^\dagger z)^2} \right] \quad (2.43)$$

$$\hat{g} = -\frac{1}{4m\rho_0} \nabla^2 + g \quad (2.44)$$

For the non Gaussian part it holds  $\mathcal{L}^{(n.G)} \sim \left(\frac{\delta\rho}{\rho_0}\right)^2$ . This part can be considered with perturbation theory. The Gaussian part on the other hand can be integrated,

$$\int \mathcal{D}\delta\rho e^{iS^{(G)}(\delta\rho, \theta, z, z^\dagger)} = \mathcal{C} \exp \left[ i \left( \int_x \rho^{(0)} J(x) + \frac{1}{2} \int_y J(x) \gamma(x-y) J(y) \right) \right], \quad (2.45)$$

where  $\gamma$  is the Greens function associated with the operator  $\hat{g}$

$$\hat{g}\gamma(x-y) = \delta(x-y). \quad (2.46)$$

$\gamma$  can be computed in momentum space. The expression can be Fourier transformed:

$$\begin{aligned} \int \frac{d\mathbf{k}^3}{(2\pi)^3} \left( -\frac{1}{4m\rho_0} \nabla_x^2 + g \right) \gamma(\mathbf{k}, x^0 - y^0) e^{i(\mathbf{x}-\mathbf{y})\mathbf{k}} &= \int \frac{d\mathbf{k}^3}{(2\pi)^3} \left( \frac{1}{4m\rho_0} \mathbf{k}^2 + g \right) \gamma(\mathbf{k}, x^0 - y^0) e^{i(\mathbf{x}-\mathbf{y})\mathbf{k}} \\ &\stackrel{!}{=} \int \frac{d\mathbf{k}^3}{(2\pi)^3} e^{i(\mathbf{x}-\mathbf{y})\mathbf{k}} \delta(x^0 - y^0) \end{aligned} \quad (2.47)$$

From this we can conclude that the effective coupling  $\gamma$  is given by

$$\gamma(\mathbf{k}, x^0 - y^0) = \frac{1}{\frac{1}{4m\rho_0} \mathbf{k}^2 + g} \delta(x^0 - y^0) = \frac{1}{g} \frac{1}{\mathbf{k}^2 / 2k_\Xi^2 + 1} \delta(x^0 - y^0) \quad (2.48)$$

with healing length

$$k_\Xi = (2m\rho_0 g)^{1/2}. \quad (2.49)$$

With this definition the effective theory can be summarized as

$$\begin{aligned} S_{\text{eff}} &= \int_x \rho^{(0)} J(x) + \frac{1}{2} \int_y J(x) \gamma(x-y) J(y) \quad \text{with} \\ \gamma(x-y) &= \int_{\mathbf{k}} \frac{1}{g} \frac{1}{\mathbf{k}^2 / 2k_\Xi^2 + 1} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \delta(t_x - t_y) \\ J(x) &= -\dot{\theta} + \frac{i}{2} \frac{z^\dagger \dot{z} - \dot{z}^\dagger z}{1 + z^\dagger z} - \frac{1}{2m} \left[ \left( \nabla\theta + \frac{i}{2} \frac{z^\dagger \nabla z - \nabla z^\dagger z}{1 + z^\dagger z} \right)^2 + \frac{(\nabla z^\dagger)(\nabla z)}{1 + z^\dagger z} - \frac{(\nabla z^\dagger)z \cdot z^\dagger(\nabla z)}{(1 + z^\dagger z)^2} \right]. \end{aligned} \quad (2.50)$$

### 2.3.3. Interpretation of the effective theory

The effective theory can be written in this compact form but it comes with two issues. First of all the resulting effective theory is non local. This doesn't come as a surprise as we integrated out a dynamical degree of freedom. If we allow for an infinite series of derivatives we can however write the theory in a local form. We write

$$\gamma(x-y) = \int_k \gamma(k) e^{ik(x-y)}. \quad (2.51)$$

## 2. Low energy effective theory

We then have

$$\begin{aligned}
\int_x \int_y \int_k J(x) \gamma(k) J(y) e^{ik(x-y)} &= \int_x \int_k \int d^d y J(x^0, \mathbf{x}) \gamma(\mathbf{k}) J(x^0, \mathbf{y}) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \\
&= \int_x \int_k \int d^d y J(x^0, \mathbf{x}) J(x^0, \mathbf{y}) \gamma(-i\nabla_x) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \\
&= \int_x \int_k \int d^d y J(x^0, \mathbf{y}) \gamma(-i\nabla_x) J(x^0, \mathbf{x}) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \\
&= \int_x J(x^0, x) \gamma(-i\nabla_x) J(x^0, x). \tag{2.52}
\end{aligned}$$

In the third equation partial integration was used. We can now expand  $\gamma(-i\nabla_x)$  in momentum space:

$$\begin{aligned}
\int_y \gamma(x-y) J(y) &= \int_y \int_k \frac{1}{g} \frac{1}{\mathbf{k}^2/2k_\Xi^2 + 1} e^{i\mathbf{k}(x-y)} \delta(x^0 - y^0) J(y) \\
&= \int_y \int_k \left[ \frac{1}{g} - \frac{\mathbf{k}^2}{2gk_\Xi^2} + \mathcal{O}\left(\frac{\mathbf{k}^4}{k_\Xi^4}\right) \right] e^{i\mathbf{k}(x-y)} \delta(x^0 - y^0) J(y) \tag{2.53}
\end{aligned}$$

The effective theory should be considered at momenta sufficiently below the healing length. In this case we see that it is sufficient to only consider the first terms of the coupling, i.e

$$\gamma(-i\nabla_x) \approx \left( \frac{1}{g} + \frac{\nabla_x^2}{4m\rho_0 g^2} \right) \tag{2.54}$$

The first terms of such an expansion are given by

$$\begin{aligned}
\mathcal{L}_{\text{eff}} \approx & \rho_0 \left( -\dot{\theta} + \frac{i}{2} \frac{z^\dagger \dot{z} - \dot{z}^\dagger z}{1 + z^\dagger z} \right) - \frac{\rho_0}{2m} \left( \nabla\theta + \frac{i}{2} \frac{z^\dagger \nabla z - \nabla z^\dagger z}{1 + z^\dagger z} \right)^2 \\
& - \frac{\rho_0}{2m} \left( \frac{\nabla z^\dagger \nabla z}{1 + z^\dagger z} - \frac{(\nabla z^\dagger z)(z^\dagger \nabla z)}{(1 + z^\dagger z)^2} \right) + \frac{1}{2g} \left( -\dot{\theta} + \frac{i}{2} \frac{z^\dagger \dot{z} - \dot{z}^\dagger z}{1 + z^\dagger z} \right)^2 + \text{h.o.t.} \tag{2.55}
\end{aligned}$$

This result can be compared to [11], where the same effective theory appears except for a sign in front of the last term. Note that in the terms up to this order only the last term comes effectively from density fluctuations. It should be valid in cases where  $\delta\rho/\rho_0 \approx 0$ . This approximation in momentum space may be later used to evaluate Feynman diagrams. Nevertheless it should be noted that we could write the effective action in momentum space, where the infinite series of derivatives  $\gamma(-i\nabla_x)$  can be resummed to an effective coupling. This should be taken into consideration if above approximation to second order derivatives doesn't lead to the expected results.

The second questionable point about the effective theory is the appearance of fields in the denominator, namely  $\frac{1}{1+z^\dagger z}$ . This arises because the Goldstone fields parametrize the surface of a sphere. One can think of different ways to solve this problem. The constraint could be implemented using a Lagrange multiplier, or by re-quantizing the theory in a careful way that takes the constraint into account. We however decided for now to understand this constraint by expanding the denominator.

$$\frac{1}{1 + z^\dagger z} \approx 1 - z^\dagger z + \mathcal{O}(\|z\|^4) \tag{2.56}$$

and interpret it as interactions of the Goldstone fields with each other. However it is hard to justify that fluctuations of  $z^\dagger z$  are small, since the potential doesn't rise with increasing values of  $z^\dagger z$ , and because of the structure of the parametrization it could even get arbitrarily large. In addition the large  $N$  expansion which is conveniently employed in this context seems to fail, as higher order terms of expansion (2.56) lead to diagrams with arbitrarily many traces, each scaling with  $N$ . This problem cannot be evaded when defining a renormalized coupling  $\lambda = gN$  while keeping the healing length fixed, since in this case even each vertex contributes a factor of  $N$ . Moreover we would rather expect the denominator to scale with  $N^{-1}$ , while the expansion gives for obvious reasons the geometric

series of terms scaling with  $N + N^2 + \dots$ . The possibility to expand the denominator goes in as an assumption from now on. Using the approximations (2.54) and (2.56) the effective action can be written in the following way.

$$\begin{aligned}
 S_{\text{eff}} \approx \int_x \left\{ z^\dagger(x) \left( \rho_0 i \partial_{x^0} + \frac{\rho_0 \nabla_x^2}{2m} \right) z(x) + \theta(x) \left( -\frac{1}{2g} \partial_{x^0}^2 - \frac{\nabla_x^2}{4k_{\Xi}^2 g} \partial_{x^0}^2 + \frac{\rho_0}{2m} \nabla_x^2 \right) \theta(x) \right. \\
 + \int_{k_1, k_2, k_3} \frac{\alpha(k_1, k_2, k_3)}{3!} \theta(k_1) \theta(k_2) \theta(k_3) e^{ix(k_1+k_2+k_3)} \\
 + \int_{k_1, k_2, k_3, k_4} \frac{\beta(k_1, k_2, k_3, k_4)}{2!2!} z_a^\dagger(k_1) z_a(k_2) z_b^\dagger(k_3) z_b(k_4) e^{ix(-k_1+k_2-k_3+k_4)} \\
 + \int_{k_1, k_2, k_3} \lambda(k_1, k_2, k_3) \theta(k_1) z_a^\dagger(k_2) z_a(k_3) e^{ix(k_1-k_2+k_3)} \\
 \left. + \int_{k_1, k_2, k_3, k_4} \frac{\mu(k_1, k_2, k_3, k_4)}{2} \theta(k_1) \theta(k_2) z_a^\dagger(k_3) z_a(k_4) e^{ix(k_1+k_2-k_3+k_4)} \right\} \quad (2.57)
 \end{aligned}$$

Here we defined the coupling terms:

$$\alpha(k_1, k_2, k_3) = -\frac{i\rho_0}{k_{\Xi}^2} \omega_1 \mathbf{k}_2 \mathbf{k}_3 + \text{all permutations } 1 \leftrightarrow 2, 2 \leftrightarrow 3, 1 \leftrightarrow 3 \quad (2.58)$$

$$\begin{aligned}
 \beta(k_1, k_2, k_3, k_4) = (\omega_1 + \omega_2) \left( \rho_0 + \left( \frac{1}{8g} - \frac{1}{8g} \frac{(\mathbf{k}_3 - \mathbf{k}_4)^2}{2k_{\Xi}^2} \right) (\omega_3 + \omega_4) + \frac{1}{2g} \frac{\mathbf{k}_3 \mathbf{k}_4}{2m} \right) \\
 + \frac{\mu}{m} \frac{1}{8g} (-\mathbf{k}_2 \mathbf{k}_4 + 6\mathbf{k}_1 \mathbf{k}_2 - \mathbf{k}_1 \mathbf{k}_3) + \text{all permutations } 1 \leftrightarrow 3 \text{ and } 2 \leftrightarrow 4 \quad (2.59)
 \end{aligned}$$

$$\lambda(k_1, k_2, k_3) = \omega_1 (\omega_2 + \omega_3) \left( \frac{i}{2g} - \frac{i}{4g} \frac{(\mathbf{k}_3 - \mathbf{k}_4)^2}{2k_{\Xi}^2} + \frac{i}{4g} \frac{k_1^2}{2k_{\Xi}^2} \right) \quad (2.60)$$

$$\mu(k_1, k_2, k_3, k_4) = \frac{1}{2gm} \omega_1 \mathbf{k}_2 (\mathbf{k}_3 + \mathbf{k}_4) - \frac{1}{4gm} \mathbf{k}_1 \mathbf{k}_2 (\omega_3 + \omega_4) + 1 \leftrightarrow 2 \quad (2.61)$$

The computation of these terms can be found in Appendix C.3. The justification of both expansions lies in the idea that the Lagrangian can be build up from the most simple terms respecting some chosen symmetry. This concept can also be used to explicitly construct an effective Lagrangian alone from symmetry considerations. In our case we should be able to find the effective Lagrangian from the Galilean symmetry, the  $U(N)$  symmetry and knowledge of the coset space, so the explicit symmetry breaking behavior. A discussion of such a derivation is done in [11].

Especially important for us is the appearance of the vertex of four  $z$  fields. This vertex should be directly related to the four-vertex of "Goldstone-modes" in [1]. We will later make some speculations which indicate that we describe the same interaction. Furthermore a vertex with an odd number of  $z$  fields is forbidden due to a charge conservation. It was demonstrated nicely in [1] that a three-vertex of such Goldstone modes, even if it appears, can be neglected. This becomes explicit in our choice of parametrization.



### 3. Applications

#### 3.1. General aspects of the low energy effective theory

At first sight the effective theory might seem unnecessarily complicated. However the approach has some significant advantages, especially for treating low energy effective theories systematically. All we need to do is to find a proper parametrization of the coset space  $G/H$ . Many quotients of manifolds are known already. Once the parametrization is found the route to an effective theory is straight forward. In our case only one integration via the path integral is required, which can be justified easily. In addition no diagonalization of the kernel is needed. The reason for this is the Lagrangian which is already parametrized in Goldstone fields. This is especially helpful when Goldstone modes are unknown in the beginning and a diagonalization of the curvature tensor isn't obvious due to a more complicated structure of the Lagrangian. On the other hand the density phase representation used in [1] follows a physical intuition and allows one to find the behavior of massless modes in terms of quantities with a clear physical interpretation. In our case it is difficult to visualize the free particle modes, while the interpretation as relative phase excitations is clear from the density phase representation. Additionally the previous approach clarifies the appearance of the Bogoliubov mode, which can be directly considered when parametrizing the coset space  $U(N)/U(N-1)$ . Thus the two approaches can be understood as complementary, one being more physical and intuitive, while the other is more systematic.

In the following the effective theory is used to understand the number of modes one can expect from Goldstone's theorem in the non relativistic theory. Afterwards we will make speculations of further applications, for which the low energy effective theory might be insightful, but where the full computation is out of scope for this thesis.

#### 3.2. Understanding Goldstones theorem in nonrelativistic theories

The main motivation for this thesis was the question how many Goldstone modes can be expected in the  $O(N)$  symmetric Gross-Pitaevskii model. On the one side the answer to the question, why we don't have more than  $N$  Goldstone modes, appears to be logical. Because our Lagrangian yields the nonlinear Schrödinger equation as equations of motion, which is of first order in the time derivative, real and imaginary part of  $\varphi_a$  are determined by just two initial conditions. They should effectively not describe independent degrees of freedom. It would then be rather a surprise if  $z$  and  $z^\dagger$  would be independent from each other. On the other hand it is a delicate topic to proof this rigorously in our context. Let us first gain some intuition that it is sufficient to analyze only  $N-1$  free particle modes as was done so in [1]. When components in both parameterizations are set equal we obtain:

$$\sqrt{\frac{\rho}{1+z^\dagger z}} e^{i\theta} |z_a| e^{i\epsilon} = \sqrt{\rho_a} e^{i\theta_a} \quad (3.1)$$

which implies

$$\epsilon = \theta_a - \theta \quad (3.2)$$

$$\sqrt{\frac{|z_a|^2}{1+z^\dagger z}} = \sqrt{\frac{\rho_a}{\rho}}. \quad (3.3)$$

We see that the normalized variables  $z$  describe a relative phase excitation times a relative density fluctuation. Imagine now that density fluctuations in all components are integrated out. This of course requires that the fluctuation of the absolute value of  $z$  is small. We then see that the degrees of freedom  $z_a$  and  $z_a^\dagger$  both depend on a single real parameter  $\epsilon$ . Due to this only the relative phase  $\epsilon$  appears in this case and the  $2N-2$  degrees of freedom with quadratic dispersion relation reduce

### 3. Applications

to  $N - 1$  relative phase excitations just as in [1]. Let's now investigate if this still holds in the case where density fluctuations are considered.

#### 3.2.1. Constraint phase space

It is a difficult topic how to count degrees of freedom in Quantum Field theory because different definitions are possible. The question can be analyzed in the phase space of the system. Usually we can start in the phase space at any point consisting of variables and conjugate momenta. If the conjugated momentum is a fixed function of the variables this is not possible; the effective phase space is a lower dimensional manifold. We can then say that the degrees of freedoms are reduced because the momenta aren't independent variables. We want to follow this idea. For this we will go back to the expansion of the effective Lagrangian in momentum space (2.55). In the paper [11] the author state from a bare scaling argument that generally speaking effective Lagrangians of this kind flow into the free fixed point in the infrared limit. It should therefore be sufficient to analyze only first order time derivative terms and neglect higher orders  $\mathcal{O}(\partial_t^2)$  which correspond to higher order energy terms. It can then even justified that only lower orders of fluctuations in fields have to be considered. Because the first time derivative of  $\theta$  comes as a total derivative term to the Lagrangian we need to include the term of next order  $\frac{1}{2g}\dot{\theta}^2$ . With this in mind we only have a closer look at

$$\mathcal{L}_{\text{eff}} \approx \rho_0 \left( -\dot{\theta} + \frac{i}{2} \frac{z^\dagger \dot{z} - \dot{z}^\dagger z}{1 + z^\dagger z} \right) + \frac{1}{2g} \dot{\theta}^2 - V(\theta, z^\dagger z) \quad (3.4)$$

with

$$V(\theta, z^\dagger z) = -\frac{\rho_0}{2m} \left( \nabla\theta + \frac{i}{2} \frac{z^\dagger \nabla z - \nabla z^\dagger z}{1 + z^\dagger z} \right)^2 - \frac{\rho_0}{2m} \left( \frac{\nabla z^\dagger \nabla z}{1 + z^\dagger z} - \frac{(\nabla z^\dagger z)(z^\dagger \nabla z)}{(1 + z^\dagger z)^2} \right) \quad (3.5)$$

The conjugate momenta are given by taking the derivative with respect to  $\dot{z}$ ,  $\dot{z}^\dagger$  and  $\dot{\theta}$ .

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{z}_a} = \frac{i}{2} \frac{z_a^\dagger}{1 + z^\dagger z} \quad (3.6)$$

$$\pi_a^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{z}_a^\dagger} = -\frac{i}{2} \frac{z_a}{1 + z^\dagger z} \quad (3.7)$$

$$\kappa = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -\rho_0 + \frac{1}{g} \dot{\theta} \quad (3.8)$$

The conjugate momenta of  $z$  and  $z^\dagger$  are independent of the "velocities"  $\dot{z}$  and  $\dot{z}^\dagger$ , and hence can't be solved for them. The corresponding Hamiltonian density turns out to be

$$\mathcal{H}(\theta, z, z^\dagger, \pi, \pi^\dagger, \kappa) = -\frac{1}{g} \kappa^2 + V(\theta, z, z^\dagger) \quad (3.9)$$

independent of the conjugate momenta of the goldstone fields. Our phase space doesn't contain  $4N - 2$  variables, but is constrained to a  $2N$  dimensional subspace. If we want to construct a Hamiltonian in the full phase space which yields the correct equations of motion we have to put this constraint in as a so called first order constraint. In the following we are going to say that two functions  $f, g$  are weakly equal,  $f \cong g$ , if they are equal under the fulfillment of a given constraint. Our primary constraints are

$$\begin{aligned} \psi_a^1 &= \pi_a - \frac{i}{2} \frac{z_a^\dagger}{1 + z^\dagger z} \\ \psi_a^2 &= \pi_a^\dagger + \frac{i}{2} \frac{z_a}{1 + z^\dagger z}. \end{aligned} \quad (3.10)$$

These constraints define the subspace in phase space we are constraint to. Remembering the famous formula

$$\dot{q} = \frac{\partial H}{\partial p} \quad (3.11)$$



we introduce a new Hamiltonian

$$\mathcal{H}' = \mathcal{H} + \dot{z}_a \psi_a^1 + \dot{z}_a^\dagger \psi_a^2 \cong \mathcal{H}. \quad (3.12)$$

At this point the velocities are unknown functions that need to be determined from additional conditions. The Hamiltonian  $\mathcal{H}'$  is a function on the full phase space. Because the constraint should fulfill

$$\psi^j \cong 0 \implies \dot{\psi}^j \cong 0 \quad (3.13)$$

we have additionally

$$\begin{aligned} \dot{\psi}_a^1 &= \{\psi_a^1, \mathcal{H}'\}_{PB} = \frac{\partial \psi_a^1}{\partial z_b} \frac{\partial \mathcal{H}'}{\partial \pi_b} + \frac{\partial \psi_a^1}{\partial z_b^\dagger} \frac{\partial \mathcal{H}'}{\partial \pi_b^\dagger} - \frac{\partial \psi_a^1}{\partial \pi_b} \frac{\partial \mathcal{H}'}{\partial z_b} - \frac{\partial \psi_a^1}{\partial \pi_b^\dagger} \frac{\partial \mathcal{H}'}{\partial z_b^\dagger} \\ &= \frac{i}{2} \frac{z_a^\dagger z_b^\dagger}{(1+z^\dagger z)^2} \dot{z}_b - \frac{i}{2} \frac{\delta_{ab}(1+z^\dagger z) - z_a^\dagger z_b}{(1+z^\dagger z)^2} \dot{z}_b^\dagger - \frac{\partial V}{\partial z_a} \cong 0, \end{aligned} \quad (3.14)$$

$$\dot{\psi}_a^2 = -\frac{i}{2} \frac{z_a z_b}{(1+z^\dagger z)^2} \dot{z}_b^\dagger + \frac{i}{2} \frac{\delta_{ab}(1+z^\dagger z) - z_a z_b^\dagger}{(1+z^\dagger z)^2} \dot{z}_b - \frac{\partial V}{\partial z_a^\dagger} \cong 0. \quad (3.15)$$

The two conditions are the Schrödinger like equations describing the time evolution of  $z$  and  $z^\dagger$ . They ensure that if we choose an initial state on the constraint surface we also stay on the constraint surface. They are identical apart from a complex conjugation and coupled. The intermediate term

$$h(\partial_{z_a}, \partial_{z_b^\dagger}) = \frac{\delta_{ab}(1+z^\dagger z) - z_a z_b^\dagger}{(1+z^\dagger z)^2} \quad (3.16)$$

is the Fubini-Study metric on the complex projective space. Special attention should be given to  $\kappa$ . If we determine the conjugate momentum in the original theory (2.27) we find

$$\kappa' = -\rho_0 - \delta\rho \quad (3.17)$$

independent of  $\dot{\theta}$ . This means as before that the phase space is reduced in dimension by the constraint that  $\kappa = -\rho$ . We see here that there is no independent gapped Higgs mode. The longitudinal heavy mode  $\delta\rho$  describes one degree of freedom together with  $\theta$ . In the effective theory this is now reflected indirectly by the fact that the Lagrangian isn't of first order in the time derivative of  $\theta$ . The density fluctuations appear again as another factor of  $\dot{\theta}$ .

### 3.2.2. Unified description of Nambu Goldstone bosons without Lorentz-invariance

This idea can be widely generalized, and was done so by Watanabe and Murayama in 2012 [2]. In non relativistic theories we usually study low energy scales at which interactions that involve the creation of massive particles and quasiparticles are strongly suppressed. Because of this it is reasonable that the massless Nambu Goldstone bosons dominate processes at these energy scales. Despite of this the number and dispersion relations of Nambu Goldstone bosons so far have been studied case by case. Watanabe and Murayama prove that the number of Nambu Goldstone Bosons  $n_{\text{NGB}}$  is given by

$$n_{\text{NGB}} = \dim G/H - \frac{1}{2} \text{rank} \rho \quad (3.18)$$

$$i\rho_{ab} = \langle \Omega | [Q_a, j_b^0(0)] | \Omega \rangle \quad (3.19)$$

both in the relativistic and in the non relativistic case.  $Q$  and  $j$  are the conserved Noether charge and current, defined as in (1.6). In the Lorentz invariant case we recover Goldstone's theorem. This theorem is proven by applying similar ideas we followed for an explicit theory. A general non relativistic effective Lagrangian has the form

$$\begin{aligned} \mathcal{L} &= c_a(\pi) \dot{\pi}^a + \frac{1}{2} \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - \frac{1}{2} g_{ab}(\pi) \nabla \pi^a \nabla \pi^b + \dots \\ &= \partial_{\pi_b} c_a(\pi)|_{\pi=0} \pi^b \dot{\pi}^a + \frac{1}{2} \bar{g}_{ab}(0) \dot{\pi}^a \dot{\pi}^b - \frac{1}{2} g_{ab}(0) \nabla \pi^a \nabla \pi^b + \mathcal{O}(\pi^3). \end{aligned} \quad (3.20)$$

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Compare to [11] and [2]. It can then be shown that the matrix  $\rho$  is proportional to the prefactor  $\sigma_{ab} = \partial_{\pi_b} c_a(\pi)|_{\pi=0}$ . In the Lorentz invariant case it vanishes, while in a non relativistic theory  $\sigma$  can contain symplectic blocks, making two degrees of freedom canonically conjugate variables. This means that the independent degrees of freedom reduce by  $1/2\text{rank}\rho$ .

### 3.3. Analyzing non-equilibrium dynamics

**Purpose of the closing section** The desired analysis of non equilibrium dynamics has created an entire research field and theory around it. Many details regarding this topic can be found in [12]. In the most recent development [1] the effective degrees of freedom were used to analyze non equilibrium behavior. It was however not easily possible to study the role and interaction of the Bogoliubov modes. In this section we want to look at the applicability of the effective theory to study non equilibrium physics. This section was mainly motivated by the hope that the Bogoliubov mode could be incorporated in a scaling analysis close to a non thermal fixed point, due to transparent interactions with the free particle modes. In order to do this the scaling of couplings in (2.57), the shape of on-shell spectral functions were studied. However several aspects of the theory make a scaling analysis hard. In the following, parts of theory and notations are given. Afterwards results that were obtained in this task so far are presented which might be useful for a more detailed study in future. Finally the sunset diagram is studied under the assumption of local couplings, using the action (2.57). However the technical difficulties indicate that a different approach might have to be taken where the manifold structure of field space is taken seriously.

#### 3.3.1. Kadanoff-Baym equations

Our aim is to derive (quantum) Boltzmann equations which describe the time evolution of occupation number distributions. For this I'll sketch the most important steps, some pieces on the other hand are left out. I refer to the literature [12] and helpful papers [16] in these cases.

Remember the definition of the effective action  $\Gamma[\phi]$  which in this context is also called *1PI* effective action. Similar to that we add another source  $R$  connected to the full propagator  $G$ . Completely equivalently to section 1.2.2 we perform a Legendre transform with respect to  $R$  to arrive at the *2PI* effective action  $\Gamma[\phi, G]$ . The evolution equation of the propagator  $G$  is given by

$$\left. \frac{\delta\Gamma[\phi, G]}{\delta G_{ab}(x, y)} \right|_{R=0} = 0. \quad (3.21)$$

We can express  $\Gamma$  as

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\text{Tr}_{\mathcal{C}} \ln G^{-1} + \frac{i}{2}\text{Tr}_{\mathcal{C}} G_0^{-1}G + \Gamma_2[\phi, G] + \text{const}. \quad (3.22)$$

with  $i\Gamma_2[\phi, G]$  being the sum of all two particle irreducible diagrams. Equation (3.21) leads to a self consistent equation for the propagator

$$G^{-1} = G_0^{-1} - \Sigma \quad (3.23)$$

with  $\Sigma$  being the proper self energy defined by

$$\Sigma_{ab}(x, y; \phi, G) := 2i \frac{\delta\Gamma_2[\phi, G]}{\delta G_{ab}(x, y)}. \quad (3.24)$$

Solving the self consistent equation for  $G$  leads to

$$G_{ab}(x, y) = G_{0,ab}(x, y) + \int_{wz, \mathcal{C}} G_{0,ac}(x, z) \Sigma_{cd}(z, w) G_{db}(w, y) \quad (3.25)$$

Next this expression is convoluted with the inverse of the full propagator which results in:

$$\int_{z, \mathcal{C}} G_{0,ac}^{-1}(x, z) G_{cb}(z, y) - \int_{z, \mathcal{C}} \Sigma_{ac}(x, z) G_{cb}(z, y) = \delta_{ab} \delta_{\mathcal{C}}(x - y). \quad (3.26)$$

To arrive at physically interesting equations we further decompose the propagator and self-energy. Defining

$$\text{spectral function:} \quad \rho_{ab}(x, y) = i\langle[\varphi_a(x), \varphi_b(y)]\rangle \quad (3.27)$$

$$\text{statistical propagator:} \quad F_{ab}(x, y) = \frac{1}{2}\langle\{\varphi_a(x), \varphi_b(y)\}\rangle \quad (3.28)$$

the propagator can be expressed as

$$G_{ab}(x, y) = F_{ab}(x, y) - \frac{i}{2}\rho_{ab}(x, y) \text{sgn}_{\mathcal{C}}(x^0 - y^0). \quad (3.29)$$

The proper self-energy in general has a local part, contributing to a mass-shift, and a nonlocal part:

$$\Sigma_{ab}(x, y) = -i\Sigma_{ab}^{(0)}(x)\delta_{\mathcal{C}}(x - y) + \bar{\Sigma}_{ab}(x, y) \quad (3.30)$$

The non local part is treated as the propagator:

$$\bar{\Sigma}_{ab}(x, y) = \Sigma_{ab}^F(x, y) - \frac{i}{2}\Sigma_{ab}^\rho(x, y) \text{sgn}_{\mathcal{C}}(x^0 - y^0). \quad (3.31)$$

To continue we want to plug in the free propagator, which we usually know. It is easily accessible from the Lagrangian. Writing the differential operator of the Gaussian terms in the action as  $\Delta_x$ , the inverse free propagator is defined as

$$iG_{0,ab}^{-1}(x, y; \phi) = \frac{\delta^2 S}{\delta\phi_a(x)\delta\phi_b(y)} = (\Delta + m^2(x))\delta_{ab}\delta_{\mathcal{C}}(x - y). \quad (3.32)$$

$m(x)$  is a local function and collects functional derivatives of non Gaussian terms. Here it is important that the Lagrangian is local, because we want to neglect such contributions later on as higher order terms in the gradient expansion. Plugging this into (3.26) in the absence of sources we obtain after a further treatment of above decompositions, which are theory independent, the Kadanoff-Baym equations.

$$(\Delta_{x,ab} + m^2(x) + \Sigma_{ab}^{(0)}(x)) F_{cb}(x, y) = - \int_0^{x^0} dz \Sigma_{ac}^\rho(x, z) F_{cb}(z, y) + \int_0^{y^0} dz \Sigma_{ac}^F(x, z) \rho_{cb}(z, y) \quad (3.33)$$

$$(\Delta_{x,ab} + m^2(x) + \Sigma_{ab}^{(0)}(x)) \rho_{cb}(x, y) = - \int_0^{x^0} dz \Sigma_{ac}^\rho(x, z) \rho_{cb}(z, y). \quad (3.34)$$

It is convenient to define the mass term  $M_{ab}^2(x) = m^2(x) + \Sigma_{ab}^{(0)}(x)$ . In these equations we can interchange  $x$  and  $y$  and subtract the two equations. The left hand side then becomes:

$$\text{l.h.s.} = (\Delta_{x,ab} - \Delta_{y,ab} + M_{ab}(x) - M_{ab}(y)) F_{cb}(x, y) \quad (3.35)$$

and accordingly for the spectral function  $\rho$ . The next step is the so called gradient expansion. For this relative coordinates are introduced, i.e.

$$X = \frac{x + y}{2}, \quad s = x - y \quad (3.36)$$

This affects all  $d + 1$  coordinates of the space-time vectors  $x, y$ . In a later step a gradient expansion in the new coordinates is employed. This results in

$$M_{ab}^{(0)}(x) - M_{ab}^{(0)}(y) = M(X + s/2) - M(X - s/2) \approx \frac{\partial M}{\partial X^\mu} s^\mu \quad (3.37)$$

which goes in as a higher order term and is neglected for this reason. This means that in the following the local contributions can be neglected. We now need to have a closer look at the equations of motion of the free particle modes ( $z$ ) and Bogoliubov modes ( $\theta$ ). Our aim is to express the equations of motion in terms of the new relative coordinates.

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#### 3.3.2. Transport equations in Wigner space

We will go back to (2.57) from which we can read off

$$\mathcal{L}_{\text{eff}}^{(G)} = z^\dagger \left( \rho_0 i \partial_{x^0} + \frac{\rho_0 \nabla_x^2}{2m} \right) z + \theta \left( -\frac{1}{2g} \partial_{x^0}^2 - \frac{\nabla_x^2}{8m\rho_0 g^2} \partial_{x^0}^2 + \frac{\rho_0}{2m} \nabla_x^2 \right) \theta \quad (3.38)$$

The  $\theta$  part is in fact an approximation of the Bogoliubov dispersion multiplied with the effective coupling to quadratic order in  $\mathbf{k}$ . To obtain the  $\Delta$  we need to take some care.  $\theta$  is a real scalar field, so the computations go through as usual.  $z$  on the other hand is a complex scalar field. So the propagators are then defined as:

$$iZ_{0,ab}^{-1}(x, y, z, z^*) := \frac{\delta^2 S}{\delta z_a^*(y) \delta z_b(x)} = \left( \rho_0 i \partial_{x^0} + \frac{\rho_0 \nabla_x^2}{2m} + m_1^2(x) \right) \delta_{ab} \delta_C(x - y) \quad (3.39)$$

$$i\Theta_0^{-1}(x, y, \theta) := \frac{\delta^2 S}{\delta \theta(y) \delta \theta(x)} = \left( -\frac{1}{2g} \partial_{x^0}^2 - \frac{\nabla_x^2}{8m\rho_0 g^2} \partial_{x^0}^2 + \frac{\rho_0}{2m} \nabla_x^2 + m_2^2(x) \right) \delta_C(x - y) \quad (3.40)$$

$m_1, m_2$  are local contributions coming from all terms that are higher than quadratic order in fields. We need to consider the two types of modes seperately and obtain for our  $\Delta$ :

$$\Delta^{(z)} = \rho_0 i \partial_{x^0} + \frac{\rho_0 \nabla_x^2}{2m} \quad (3.41)$$

$$\Delta^{(\theta)} = -\frac{1}{2g} \partial_{x^0}^2 - \frac{\nabla_x^2}{8m\rho_0 g^2} \partial_{x^0}^2 + \frac{\rho_0}{2m} \nabla_x^2. \quad (3.42)$$

Note that the differential operator for free particle fields is of first order in the time component, which gives a minus sign when exchanging  $x^0$  and  $y^0$ . The operators then are given in relative coordinates by:

$$\begin{aligned} \mathcal{D}^{(z)} &:= \Delta_x^{(z)} - \Delta_y^{(z)} = \rho_0 i \frac{\partial}{\partial X^0} + \frac{\rho_0 2 \nabla_X \cdot \nabla_s}{2m} \\ \mathcal{D}^{(\theta)} &:= \Delta_x^{(\theta)} - \Delta_y^{(\theta)} = -\frac{1}{2g} 2 \frac{\partial}{\partial X^0} \frac{\partial}{\partial s^0} - \frac{1}{8m\rho_0 g^2} \left[ 2 \frac{\partial}{\partial X^0} \frac{\partial}{\partial s^0} \left( \frac{1}{4} \nabla_X^2 + \nabla_s^2 \right) \right. \\ &\quad \left. + 2 \nabla_X \nabla_s \left( \frac{1}{4} \frac{\partial^2}{\partial X^0} + \frac{\partial^2}{\partial s^0} \right) \right] + \frac{\rho_0}{2m} 2 \nabla_X \cdot \nabla_s. \end{aligned} \quad (3.43)$$

For the next step we go to Wigner space. This is done by a Fourier transform with respect to  $s$ . We will further assume  $s^0$  ranging from  $-\infty$  to  $\infty$ , which restricts the description being valid to describe the system after a sufficiently late time after the initial state was prepared. As we fixed the initial time to zero we have  $x^0 \geq 0, y^0 \geq 0$ , such that  $s^0$  ranges from  $0 - y^0$  to  $x^0 + 0$ , which in terms of the relative coordinates means that  $s_0 \in [-2X_0, 2X_0]$ . Consequently

$$\begin{aligned} \tilde{F}(X, p) &= \int_{-2X^0}^{2X^0} ds^0 e^{i\omega s^0} \int_{-\infty}^{\infty} d^d s e^{-i\mathbf{p}\mathbf{s}} F \left( X + \frac{s}{2}, X - \frac{s}{2} \right) \\ \tilde{\rho}(X, p) &= \int_{-2X^0}^{2X^0} ds^0 e^{i\omega s^0} \int_{-\infty}^{\infty} d^d s e^{-i\mathbf{p}\mathbf{s}} \rho \left( X + \frac{s}{2}, X - \frac{s}{2} \right) \end{aligned} \quad (3.44)$$

Then the limit  $X^0 \rightarrow \infty$  is taken. The differential operators (3.43) can also be written in Wigner space which gives

$$\begin{aligned} \mathcal{D}^{(z)}(X, p) &= \rho_0 i \frac{\partial}{\partial X^0} - \frac{\rho_0 i \nabla_X \cdot \mathbf{p}}{m} \\ \mathcal{D}^{(\theta)}(X, p) &= -\frac{i}{g} \frac{\partial}{\partial X^0} p^0 - \frac{i}{4m\rho_0 g^2} \left[ \frac{\partial}{\partial X^0} p^0 \left( \frac{1}{4} \nabla_X^2 - \mathbf{p}^2 \right) - \nabla_X \mathbf{p} \left( \frac{1}{4} \frac{\partial^2}{\partial X^0} - p^{02} \right) \right] - \frac{i\rho_0}{m} \nabla_X \cdot \mathbf{p} \end{aligned} \quad (3.45)$$

The idea of the gradient expansion is to neglect all terms of order  $\mathcal{O}(\partial_X \partial_p)$  in Wigner space. As we don't have a source of  $\partial_p$  terms, we keep all terms with only one derivative in  $X$ . The right hand side of (3.33) can be treated independently of the theory. The corresponding calculations can be found in [17]. Before we write down the transport equations using (3.45) we assume spatial homogeneous initial conditions. This means that  $G(x, y) = G(x_0, y_0, \mathbf{x} - \mathbf{y}) = G(x_0, y_0, \mathbf{s})$  for all propagators  $G$  and for both modes. In this case  $\tilde{F}$  and  $\tilde{\rho}$  become independent of the center space coordinate  $\mathbf{X}$  and the equations reduce to:

$$\left( \frac{\mathbf{p}^2}{2k_{\Xi}^2 g} - \frac{1}{g} \right) i p_0 \partial_{X^0} \tilde{F}^{(\theta)}(X, p) = \tilde{\Sigma}_{\rho}^{(\theta)}(X, p) \tilde{F}^{(\theta)}(X, p) - \tilde{\Sigma}_F^{(\theta)}(X, p) \tilde{\rho}^{(\theta)}(X, p) \quad (3.46)$$

$$\left( \frac{\mathbf{p}^2}{2k_{\Xi}^2 g} - \frac{1}{g} \right) i p_0 \partial_{X^0} \tilde{\rho}^{(\theta)}(X, p) = 0 \quad (3.47)$$

$$2\rho_0 i \partial_{X^0} \tilde{F}_{ab}^{(z)}(X, p) = \tilde{\Sigma}_{\rho}^{(z)}(X, p) \tilde{F}^{(z)}(X, p) - \tilde{\Sigma}_F^{(z)}(X, p) \tilde{\rho}^{(z)}(X, p) \quad (3.48)$$

$$2\rho_0 i \partial_{X^0} \tilde{\rho}_{ab}^{(z)}(X, p) = 0 \quad (3.49)$$

where the right hand side doesn't depend on homogenous initial conditions but also holds in the more general case. As we can see the expansion of the effective coupling  $\gamma$  is recovered for the Bogoliubov mode.  $X_0 = t$  will be treated as the time coordinate, so the equations imply a spectral function constant in time. Under this assumption we will from now on focus on (3.48).

### 3.3.3. Scaling of the on-shell spectral functions and couplings

We next need an estimate of the spectral functions  $\rho$ . It is directly related to the retarded propagator by

$$G_{ab}^R(x, y) = \rho_{ab}(x, y) \Theta(x^0 - y^0) \quad (3.50)$$

where  $\Theta$  denotes the  $\Theta$ -function. We make the ansatz for the spectral function that the quasiparticles are on shell. In this case the spectral function can be determined from the retarded propagator of the free theory. The retarded propagator differs from the free Feynman propagator only by a pole shift.

$$Z_{(0),ab}^R(x - y) = \lim_{\varepsilon \rightarrow 0^+} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{1}{\rho_0(k^0 + i\varepsilon - \mathbf{k}^2/2m)} e^{-ik(x-y)}. \quad (3.51)$$

We can use the fact that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{x - i\varepsilon} = \mathcal{P} \frac{1}{x} + i\pi \delta(x) \quad (3.52)$$

where  $\mathcal{P}$  denotes the principal value. The spectral function is now given by:

$$\begin{aligned} \tilde{\rho}^{(z)}(k) &= -2\Im[Z_{(0),ab}^R(k)] = -2\Im \left[ \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\rho_0(k^0 + i\varepsilon - \mathbf{k}^2/2m)} \right] \\ &= \frac{2\pi}{\rho_0} \delta \left( k^0 - \frac{\mathbf{k}^2}{2m} \right) \end{aligned} \quad (3.53)$$

The situation for the Bogoliubov modes is more complicated. I suggest the following approach: We look again at the effective action

$$S_{\text{eff}} = \int_x \left[ \rho^{(0)} J(x) + \frac{1}{2} \int_y J(x) \gamma(x - y) J(y) \right] \quad (3.54)$$

This time we don't expand the effective coupling  $\gamma$  in powers of  $\mathbf{k}$ . The gaussian terms are then still local due to Parseval's rule. In Fourier space the Gaussian terms then read:

$$\rho_0 \left( k^0 - \frac{\mathbf{k}^2}{2m} \right) z^\dagger(k) z(k) - \frac{1}{2} \gamma(k) \left[ k^{02} - \frac{\mathbf{k}^2}{2m} \left( \frac{\mathbf{k}^2}{2m} + 2g\rho^{(0)} \right) \right] \theta(k) \theta(-k) \quad (3.55)$$

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with

$$\gamma(k) = \frac{1}{g} \frac{1}{\mathbf{k}^2/2k_{\Xi}^2 + 1}. \quad (3.56)$$

This again gives the retarded propagator of the Bogoliubov mode:

$$\Theta_{(0),ab}^R(x, y) = \lim_{\varepsilon \rightarrow 0^+} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{2g(\mathbf{k}^2/2k_{\Xi}^2 + 1)}{\left[(k^0 + i\varepsilon)^2 - \frac{\mathbf{k}^2}{2m} \left(\frac{\mathbf{k}^2}{2m} + 2g\rho^{(0)}\right)\right]} e^{-ik(x-y)}. \quad (3.57)$$

The denominator can be treated with a partial fraction decomposition. Define  $\omega(\mathbf{k}) = \sqrt{\frac{\mathbf{k}^2}{2m} \left(\frac{\mathbf{k}^2}{2m} + 2g\rho^{(0)}\right)}$ .

The denominator has the poles at:

$$k^0 = \pm\omega(\mathbf{k}) - i\varepsilon. \quad (3.58)$$

Writing:

$$\frac{1}{(k^0 - i\varepsilon)^2 - \omega(\mathbf{k})^2} = \frac{A}{k^0 - \omega(\mathbf{k}) + i\varepsilon} + \frac{B}{k^0 + \omega(\mathbf{k}) + i\varepsilon} \quad (3.59)$$

yields

$$A + B = 0 \quad (3.60)$$

$$-2B\omega(\mathbf{k}) = 1 \quad (3.61)$$

such that

$$\frac{1}{(k^0 + i\varepsilon)^2 - \omega(\mathbf{k})^2} = \frac{1}{2\omega(\mathbf{k})} \left( \frac{1}{k^0 - \omega(\mathbf{k}) + i\varepsilon} - \frac{1}{k^0 + \omega(\mathbf{k}) + i\varepsilon} \right). \quad (3.62)$$

Using the same steps as before on both terms we finally obtain the spectral function:

$$\tilde{\rho}^{(\theta)}(k) = \frac{2\pi\tilde{g}(k)}{\omega(\mathbf{k})} [\delta(k^0 - \omega(\mathbf{k})) - \delta(k^0 + \omega(\mathbf{k}))] \quad (3.63)$$

At this stage however we are unable to employ the approximation of the effective coupling  $\tilde{g} = \gamma^{-1}$ . To obtain the approximation that we made for the interaction terms it is reasonable to expand the fraction  $\frac{\tilde{g}(k)}{\omega(\mathbf{k})}$  in a Laurent series. Note that  $\tilde{g}(k)$  doesn't actually depend on  $k^0$ .

$$\left(\frac{g(k)}{\omega(\mathbf{k})}\right)^2 = \frac{2gm}{\rho_0} \frac{1}{\mathbf{k}^2} + \mathcal{O}(\|k\|^0) \quad (3.64)$$

This determines the scaling of  $\rho^\theta$ . We then have:

$$\rho^\theta \sim \frac{1}{\rho_0} \frac{\sqrt{2}k_{\Xi}}{\|\mathbf{k}\|} \quad (3.65)$$

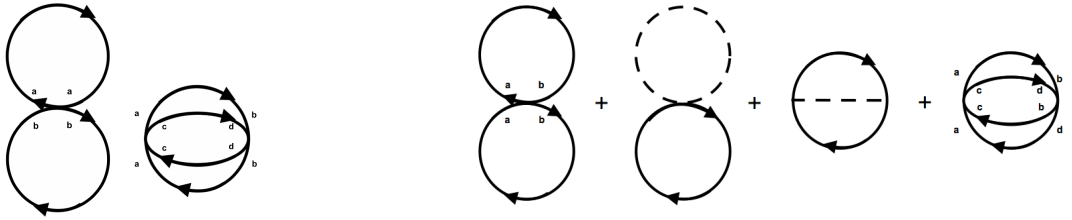
Alternatively we can start with the free propagator of the action at order  $\mathbf{k}^2$ . To proof consistency it would be useful to show the same scaling of the spectral function also in this case.

The scaling of the couplings can be read of the effective action (2.57). For this we will employ the on-shell approximation. As they don't scale with a single scaling exponent we have to take the smallest one, results are given in (3.66). The most important one is  $\beta \sim \mathbf{k}^2$  on shell.

#### 3.3.4. Probable scaling of the sunset diagram

We now want to use previously determined quantities to make a rough estimate of the critical exponent  $\beta$ . The motivation to do this is at this point mainly to show that our approach is consistent with the previous work [9]. To derive the scaling behaviour of free particle modes  $z$  we want to use the transport equation (3.48) for which we need to have knowledge of the selfenergy of free particle modes. To obtain diagrammatic expressions for the self energy we have already employed several approximations, in particular the expansions used to obtain the approximate effective action (2.57). In the next step we need to assume that the couplings are small such that we can treat them perturbatively. A common approach to ensure this is by doing a large  $N$  expansion.

**Some words on the large  $N$  case** Usually a fixed renormalized coupling  $\lambda = gN$  is introduced. A vertex in the fundamental theory then contributes a factor  $N^{-1}$ . However here the fundamental coupling  $g$  goes in inversely, hence such a procedure would yield unreasonable factors of  $N$  and the limit  $N \rightarrow \infty$  can not be taken. It is not reasonable to do the large  $N$  expansion here, but at a heuristic level the order of diagrams stays as before: We assume that the couplings of Goldstone fields  $\alpha, \beta, \lambda, \mu$  can be treated perturbatively after fixing some large value  $N$ , which is reasonable since we study low momenta far below the healing length for which coupling function have small values. In this case the lowest order contributions to the 2PI-effective action come from vacuum diagrams with few vertices, in our case we collect all diagrams with up to two vertices. Now we sort the contributions according to the scaling with  $N$  without taking the limit. In this case the dominating contributions are to leading order and next leading order in  $N$  the same used in [1]. The justification for this is



(a) Leading order diagrams  $\sim (N - 1)^2$  up to two vertices with coupling  $\beta$ . (b) First next leading order diagrams  $\sim N - 1$  up to two vertices. (more diagrams are possible)

Figure 3.1.: First diagrams of the effective action  $\Gamma_2$  contributing to the self energy of free particle modes. Solid lines represent free particle modes, while dashed lines correspond to the Bogoliubov mode.

the fact that the coupling functions don't contain additional constants apart from  $g$  and  $\rho_0$  that could cancel a factor of the size of  $N$ . However this usually convenient approach still has the big issue with the expansion of the denominator  $(1 + z^\dagger z)^{-1}$ , which contributes higher powers of  $N$ . This has already been discussed in subsection 2.3.3. A different approach is to look at the scaling of the coupling functions with momenta. This can be done using the on shell approximation which is used afterwards anyway, which gives:

$$\alpha \sim \mathbf{k}^3, \quad \beta \sim \mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4), \quad \lambda \sim \mathbf{k}^3 + \mathcal{O}(\mathbf{k}^5), \quad \mu \sim \mathbf{k}^3 + \mathcal{O}(\mathbf{k}^4) \quad (3.66)$$

The coupling function  $\beta$  dominates in this scheme, which corresponds to the same vertex that dominates in the large  $N$  expansion. In the case of large  $N$  and low momenta it seems to be reasonable to investigate further the contributions in figure 3.1a to  $\Gamma_2$ .

**Computation of the sunset diagram with local couplings** Our aim is to derive the scaling behavior of the theory near a non thermal fixed point. This scaling behavior shall be analyzed from equation (3.48) for which we need knowledge of the self energies. The self energies can't be computed to all orders. According to our previous considerations the dominating terms are a local tadpole diagram and a non local sunset diagram, the latter leading to a critical exponent of  $\beta = 1/2$  in the previous approach [1]. The computation of Feynman rules technically difficult in our case such that I will only make an estimate, assuming that the coupling are local because the coupling functions only contain derivatives. In this case they can be taken in front of the expressions one would get from a constant coupling. The dominating three loop contribution to  $\Gamma_2[G, \phi]$  in orders of  $N$  is given by:

$$i\Gamma_2^{(3a)}[G] = - \int_{x,c} \int_{y,c} \beta_x \beta_y G_{db}(x, y) G_{ba}(y, x) G_{ca}(x, y) G_{ac}(x, y) = - \int_{x,c} \int_{y,c} \beta_x \beta_y N^2 (G(x, y) G(y, x))^2 \quad (3.67)$$

where the propagator corresponding to free particle modes is denoted by  $G_{ab} = G\delta_{ab}$ .

$$\Sigma^{(3a)}(x, y) = 2N^2 \beta_x \beta_y [2G(x, y)G(x, y)G(y, x) + 2G(x, y)G(y, x)G(y, x)] \quad (3.68)$$

### 3. Applications

This is of the same form as the contribution given in [18]. Because of the product rule exist two diagrams in opposing directions. Possible cancelations are neglected at this point and we proceed with just one of the two otherwise identical diagrams. The propagators can be decomposed

$$G(x, y)G(x, y)G(y, x) = \left( F(x, y) - \frac{i}{2}\rho(x, y) \operatorname{sgn}_{\mathcal{C}}(x^0 - y^0) \right)^2 \left( F(x, y) - \frac{i}{2}\rho(x, y) \operatorname{sgn}_{\mathcal{C}}(x^0 - y^0) \right)^* . \quad (3.69)$$

Next we use the symmetries  $F(x, y)^* = F(y, x)$  and  $\rho(x, y)^* = -\rho(y, x)$ . The terms can now be summarized and compared with the decomposition of the self energy in spectral part and statistical part. This gives:

$$\begin{aligned} \Sigma^F(x, y) &= 4N^2\beta_x\beta_y \left[ F(x, y)F(x, y)F(y, x) - \frac{1}{4}\rho(x, y)\rho(x, y)F(y, x) - \frac{2}{4}\rho(x, y)F(x, y)\rho(y, x) \right] \\ \Sigma^\rho(x, y) &= 4N^2\beta_x\beta_y \left[ F(x, y)F(x, y)\rho(y, x) + 2F(x, y)\rho(x, y)F(y, x) - \frac{1}{4}\rho(x, y)\rho(x, y)\rho(y, x) \right] . \end{aligned} \quad (3.70)$$

The shape of the expression in Wigner space stays the same because of the linearity of the Fourier transform. From here it can be proceeded as in [18] after relabeling  $l \leftrightarrow -m$  to summarize terms. In Wigner space we additionally employ the so called Kadanoff-Baym ansatz, which can be motivated by equilibrium considerations.

$$\tilde{F}(X^0, p) = \left( f(X^0, p) + \frac{1}{2} \right) \tilde{\rho}(X^0, p) \quad (3.71)$$

Plugging into (3.48) the self energies, expressed through "particle occupation numbers"  $f$  and spectral functions  $\tilde{\rho}$ , we obtain

$$\begin{aligned} \rho_0 i \partial_{X^0} f(X^0, p) \tilde{\rho}^Z(X, p) &= \int_{\mathbf{k}} \int_{\mathbf{l}} \int_{\mathbf{m}} 4N^2\beta(\mathbf{k} + \mathbf{l} - \mathbf{m} - \mathbf{p})\beta(\mathbf{p} - (\mathbf{k} + \mathbf{l} - \mathbf{m}))\tilde{\rho}_{\mathbf{k}}\tilde{\rho}_{\mathbf{l}}\tilde{\rho}_{-\mathbf{m}}\tilde{\rho}_{\mathbf{p}} \\ &\quad \times \delta(\mathbf{k} + \mathbf{l} - \mathbf{m} - \mathbf{p})(1 + f_{\mathbf{p}})(1 + f_{\mathbf{k}})f_{\mathbf{l}}f_{\mathbf{m}} - f_{\mathbf{p}}f_{\mathbf{k}}(1 + f_{\mathbf{l}})(1 + f_{\mathbf{m}}) \end{aligned} \quad (3.72)$$

We now use the on-shell approximation that was previously discussed. We have

$$\tilde{\rho}^{(z)}(\mathbf{k}) = \frac{2\pi}{\rho_0} \delta \left( k^0 - \frac{\mathbf{k}^2}{2m} \right) . \quad (3.73)$$

Integrating over  $\int d\mathbf{p}^0/(2\pi)$ , relabeling  $X^0 \rightarrow t$  and after cancelation of  $i$  the transport equation (3.48) reads to this order:

$$\begin{aligned} i \partial_t f(t, \mathbf{p}) &= \int_{\mathbf{k}} \int_{\mathbf{l}} \int_{\mathbf{m}} 4\beta(\mathbf{k} + \mathbf{l} - \mathbf{m} - \mathbf{p})\beta(\mathbf{p} - (\mathbf{k} + \mathbf{l} - \mathbf{m})) \frac{8\pi^3}{\rho_0} \delta(\omega(\mathbf{k}) + \omega(\mathbf{l}) + \omega(\mathbf{m}) - \omega(\mathbf{p})) \\ &\quad \times \delta(\mathbf{k} + \mathbf{l} - \mathbf{m} - \mathbf{p})(1 + f_{\mathbf{p}})(1 + f_{\mathbf{k}})f_{\mathbf{l}}f_{\mathbf{m}} - f_{\mathbf{p}}f_{\mathbf{k}}(1 + f_{\mathbf{l}})(1 + f_{\mathbf{m}}) \\ &= I[f] \end{aligned} \quad (3.74)$$

with the definitions  $\omega(\mathbf{k}) = \mathbf{k}^2/2m$ ,  $f_{\mathbf{k}} = f(t, \omega(\mathbf{k}), \mathbf{k})$  and  $\beta(\mathbf{k}) = \beta(\omega(\mathbf{k}), \mathbf{k})$  the scaling can now be determined. We assume the scaling hypothesis

$$f(\mathbf{p}, t) = s^{\alpha/\beta'} f(s\mathbf{p}, s^{-1/\beta'} t) , \quad (3.75)$$

where the notation  $\beta'$  was introduced to distinguish the exponent from the coupling function  $\beta$ . In the scattering integral we have to make a transformation  $\mathbf{k} \rightarrow \mathbf{k}/s$ ,  $\mathbf{l} \rightarrow \mathbf{l}/s$ ,  $\mathbf{m} \rightarrow \mathbf{m}/s$ . One momentum integral scales the opposite to the delta function in momentum vectors. We are left with a factor from the transformation rule  $s^{2d}$ . The occupation number distributions  $f$  contribute according to above scaling hypothesis a factor of  $s^{-3\alpha/\beta}$ . Note that the term quartic in  $f$  is canceled out. The energies



$\omega(\mathbf{k}) = \mathbf{k}^2/2m$  scale with a  $s^z = s^2$ . Consequently the  $\delta$ -function contributes the opposite factor. We assume that the scattering integral has a well defined scaling exponent

$$I[f](s\mathbf{k}, s^{-1/\beta'}t) = s^\mu I[f](\mathbf{k}, t). \quad (3.76)$$

The  $T$ -matrix scattering element  $T \sim \beta$  is given by

$$|T(\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{p})|^2 = 4\beta(\mathbf{k} + \mathbf{l} - \mathbf{m} - \mathbf{p})\beta(\mathbf{p} - (\mathbf{k} + \mathbf{l} - \mathbf{m}))\frac{8\pi^3}{\rho_0} \quad (3.77)$$

and has the scaling exponent  $m$  defined by

$$|T(s\mathbf{k}, s\mathbf{l}, s\mathbf{m}, s\mathbf{p})| = s^m |T(\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{p})| \quad (3.78)$$

from which we obtain

$$\mu = 2d + 2m - z - 3\frac{\alpha}{\beta'} = 2d + 2m - 2 - 3\frac{\alpha}{\beta'}. \quad (3.79)$$

The kinetic equation (3.74) now implies the relation

$$\alpha = 1 - \beta'\mu. \quad (3.80)$$

By plugging in  $\mu$  and rearranging terms we obtain

$$\beta' = \frac{1}{2(d+m) - z - 2\alpha/\beta'}. \quad (3.81)$$

$\beta'$  can now be determined with one additional condition. Assuming conservation of quasi particle number we have

$$\alpha = \beta'd \quad (3.82)$$

and in the case of energy conservation

$$\alpha'' = \beta''(d+z). \quad (3.83)$$

From the scattering integral we see that  $m$  is the scaling of  $\beta'$ , which is given to lowest order by  $m = 2$ . In addition the free particle modes have a quadratic dispersion relation, which implies  $z = 2$ . With this we obtain in the case of number conservation

$$\beta' = \frac{1}{2(d+m) - z - 2\alpha/\beta'} = \frac{1}{2d - 2d + 2 \cdot 2 - 2} = \frac{1}{2} \quad (3.84)$$

and in the case of energy conservation

$$\beta'' = \frac{1}{2(d+m) - z - 2(d+z)} = \frac{1}{2 \cdot 2 - 3 \cdot 2} = -\frac{1}{2} \quad (3.85)$$

which coincides with the results given in [1]. We can conclude that the interaction of free particle modes with each other exhibits under the assumption of a universal scaling behavior the same scaling as the "Goldstone modes" derived from the density phase representation. As additionally for small density fluctuations the free particle modes  $z$  are proportional to the fundamental fields, this scaling must translate to the fundamental fields. Above discussions are so far estimations. Several steps in the calculation require a more detailed study in future, in particular it should be checked that the assumed Feynman rules hold and it what order to consider contributing diagrams.

### 3. Applications

#### 3.3.5. The one loop self energy diagram

Additionally to the previous discussion the effect of the Bogoliubov mode on universal scaling can be studied. The contribution to the self energy of the free particle at lowest order is given by a typical one loop diagram, in structure similar to the lowest order self energy contribution of the electron self energy. The corresponding structure of the Boltzmann equation has been derived before, see for example [16]. In the scattering integral appears the  $\delta$ -function

$$\delta(\omega_z(\mathbf{k}) + \omega_\theta(\mathbf{l}) - \omega_z(\mathbf{p})). \quad (3.86)$$

However the different dispersion relations of Bogoliubov mode and free particle modes pose the problem that a clear scaling behavior can't be read off the integral. This remains an open question motivating future work to analyze which scaling behavior can be expected when the Bogoliubov mode can't be neglected.

#### 3.4. Topological properties of the parametrization

In the effective theory arises the problem with the denominators in the Lagrangian, which can't be treated in the canonical formalism. However this property is not unique to our effective theory. It is rather something one would expect, as we define our goldstone fields as parameters of the coset space, which is by definition a sub-manifold of a Euclidean space, and doesn't allow for a simple canonical treatment. Non-linear sigma models show very similar behavior. One aspect I want to delve is that some non-linear sigma models yield solutions characterized by the homotopy group of the field space. Motivated by this I want to point out the existence of non trivial homotopy groups also in our case. For this we go one step back where fundamental fields  $\varphi_a$  are parametrized as

$$\varphi_a = \sqrt{\rho} \cdot t \cdot p_a \quad (3.87)$$

with the constraint that

$$\begin{aligned} t(x) &\in S^1 \\ p(x) &\in \mathbb{C}\mathbb{P}^{N-1}. \end{aligned} \quad (3.88)$$

We split these constraints up in order to describe the Goldstone modes and Bogoliubov modes independently. Let's now assume that solutions exist that approach one single value at infinity in space, so for  $\mathbf{x} \in \mathbb{R}^d$ :

$$\begin{aligned} t(x^0, \mathbf{x}) &\rightarrow \tilde{t} \\ p(x^0, \mathbf{x}) &\rightarrow \tilde{p} \quad \text{for } \|\mathbf{x}\| \rightarrow \infty. \end{aligned} \quad (3.89)$$

In this case a theorem of algebraic topology tells us that the maps  $t$  and  $p$  can be continuously extended to  $\mathbb{R}^d \cup \infty$  for every time  $x^0$ . This set is compact and homeomorphic to a sphere, in fact

$$\mathbb{R}^d \cup \infty \cong S^d. \quad (3.90)$$

So  $t$  and  $p$  define maps

$$t : S^d \rightarrow S^1 \quad (3.91)$$

$$p : S^d \rightarrow \mathbb{C}\mathbb{P}^{N-1}. \quad (3.92)$$

This type of maps are classified by the homotopy groups of the manifolds  $S^1$  and  $\mathbb{C}\mathbb{P}^{N-1}$ . Two maps or classical field configurations are called homotopic if they can be continuously deformed into each other with time  $t \in [a, b]$ . If this is not possible the non homotopic maps are distinguished solutions and characterized by a topological charge. So we should expect solutions characterized by the elements of

$$\pi_d(S^1) \text{ and } \pi_d(\mathbb{C}\mathbb{P}^{N-1}). \quad (3.93)$$

### 3.4. Topological properties of the parametrization

The Bogoliubov field  $t$  (or previously  $\theta$ ) doesn't have such topological solutions, except if  $d = 1$ . This is also the reason why we can neglect the term  $\rho_0 \dot{\theta}$  in the effective theory. The Goldstone fields on the other hand are much more interesting. Let's consider the case  $d \geq 2$ . In this situation we have

$$\pi_d(\mathbb{C}\mathbb{P}^{N-1}) \cong \pi_d(S^{2N-1}). \quad (3.94)$$

The homotopy groups are trivial in many cases. For example can any loop on a sphere be contracted to a point in a smooth way. This is equivalent to the statement that the homotopy group  $\pi_1(S^2) = 0$ . More generally we have  $\pi_i(S^j) = 0$  for  $i < j$ . As we can see we have a lower bound on  $N$  for when the theory is definitely topologically trivial. We obtain

$$2N - 1 > d \implies N > \frac{d}{2} + \frac{1}{2}. \quad (3.95)$$

For three spatial dimensions, so  $d = 3$  the theory is topologically trivial for  $N > 2$ . In this cases we can be sure that topological charges induced by Goldstone modes have no influence on the system, unless additional constraints are enforced. For high dimension and low  $N$  on the other hand the theory might be topologically non trivial, for example:

$$\pi_3(S^3) \cong \mathbb{Z} \quad (3.96)$$

in the case of  $d = 3, N = 2$  which possibly implies the existence of topological charges in the system. If a map with constant boundary conditions exists they can not be continuously deformed over time into a solution of different topological charge. For high dimensions the homotopy groups become in fact highly non trivial, yielding a finite amount of possible topological charges. In  $d = 4$  it is for example

$$\pi_4(S^3) = \mathbb{Z}_2. \quad (3.97)$$



## 4. Conclusions

**Generalizations of the idea and solutions given by the effective theory** In this thesis we investigated a systematic approach how to study the low energy behavior of a quantum field theory exhibiting spontaneous symmetry breaking. This is done by a careful parametrization of the coset space  $U(N)/U(N-1)$ , which distinguishes massless degrees of freedom from heavy ones. The heavy modes can then be integrated out to obtain a low energy effective theory. This idea should be applicable to many similar and more complex situations.

In a theory without Lorentz invariance dispersion relations can be non linear. Therefore, it should be noted that it is useful to distinguish modes with different dispersions already in the parametrization. This has two main advantages: firstly, the behavior of one type of mode can dominate over the other which allows for further simplifications of the model. Secondly, the absence of degrees of freedom in other low energy effective models can be clarified because several modes with non linear dispersion relation can describe one degree of freedom together. If one wants to take this route a recent paper gives a unified description of Goldstone's theorem in a theory without Lorentz invariance [2].

In the case of the most recent work of our group [1] we are able to clarify that also from the perspective of Goldstone's theorem  $N$  massless degrees of freedom dominate the low energy behavior. In the case of a non thermal fixed point at low momentum scales the same critical exponents can be expected. In addition in our low energy effective theory fewer degrees of freedom than in [1] are integrated out; only one integration over a longitudinal mode is required. This integration can be justified independent of the chosen groundstate  $\phi_0$ . This gives the opportunity to understand the role of density fluctuations clearer. While the total density fluctuation  $\delta\rho$  is strongly suppressed at low momenta, relative density fluctuations between components can be large because they are described by the massless free particle mode. With respect to this the theory gives a slightly more general picture. Another important result is the absence of a three vertex of modes with free particle dispersion, which is forbidden by a  $U(1)$  symmetry in the free particle fields. This gives an additional insight why such a vertex can be neglected in the case of a density phase representation to express Goldstone fields.

The background for our description on the other side is the fact that the  $O(N)$  symmetry of the Gross-Pitaevskii model can be extended to a  $U(N)$  symmetry. Moreover the construction gives a geometrical picture of the modes and their behavior. The free particle fields can be understood as parameters of a symplectic manifold, while the Bogoliubov mode lives in the fibre  $S^1$ . The implications of this geometric picture should be studied. It appears that some homotopy groups related to the modes don't vanish. The possibility of topological charges and their effect on universality might be of interest in this cases.

**Limitations of the approach** Despite the efforts that were taken to understand different aspects of the model, like scaling of couplings and spectral functions, the model seems to offer several difficulties which make the analysis of universal behavior complicated. Couplings, the spectral function of the Bogoliubov mode and the effective coupling  $\gamma$  don't exhibit a well defined scaling behavior, hence they have to be approximated to some order. Due to this it is not insightful to expand  $\gamma$  beyond constant order since higher order terms have to be neglected later on. The different scalings become especially problematic for the interaction of Bogoliubov mode and free particle modes, where the scaling of derived scattering integrals becomes ambiguous already at low orders. Such a problem will typically arise in the quantum Boltzmann equation when modes of different dispersion appear. In addition we have to deal with the denominator  $(1+z^\dagger z)^{-1}$  appearing from the normalization. In our treatment of the theory it is expanded without deeper justification. A good argument why higher orders in fields don't give a significant contribution at the critical point is lacking so far, especially when the system is out of equilibrium. It is also questionable whether the result of  $\beta = 1/2$  backs up the approximations since it is related to first order time derivative and a second order spatial

#### 4. Conclusions

derivative in the four vertex, which both appear already in the full effective theory. On the contrary, the denominator seems to be a crucial part of this approach, since it is closely connected to the metric on field space; the Fubini-Study metric on  $\mathbb{C}\mathbb{P}^{N-1}$ . This property should be taken more seriously. The treatment of the parametrization is surely not exhausted yet and offers the possibility of related works. In this regards I want to especially emphasize the connection to non linear sigma models. Ideas coming from the related theory should be helpful for technical problems like the treatment of the denominator, which is a typical case in such models.

## A. Notations and Conventions

In this chapter I'm giving an introduction to the most important notations I used.

**Units** Throughout this thesis we'll use

$$\hbar = c = 1 \quad (\text{A.1})$$

In this thesis we always consider a system in  $d + 1$  dimensions, where  $d$  is the dimension of the space, and the additional dimension represents time. Despite being in a non relativistic system we adapt the notation of four vectors, where the components are labeled from 0 to  $d$ . The  $d + 1$  dimensional vectors is written as a normal symbol, e.g.  $x$ , while the  $d$  dimensional space vector is summarized as a bolt symbol, e.g.  $\mathbf{x}$ :

$$x = \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix}, \quad k = \begin{pmatrix} k^0 \\ \mathbf{k} \end{pmatrix} \quad (\text{A.2})$$

We label spacetime points by

$$x, y, z, w \quad (\text{A.3})$$

and points in momentum space as

$$k, l, m, p \quad (\text{A.4})$$

The zero component of points in momentum space represents the angular frequency or energy. We further adopt the pseudo dot product induced by the Minkowskian metric in  $d + 1$  dimensions, i.e.

$$kx = k^0 x^0 - \mathbf{k} \cdot \mathbf{x}. \quad (\text{A.5})$$

**Fouriertransforms** For Fourier transforms we use the convention that

$$\mathcal{F}[f](k) = \int_{\mathbb{R}^{d+1}} d^{d+1}x f(x) e^{ikx} \quad (\text{A.6})$$

and

$$\mathcal{F}^{-1}[f](x) = \int_{\mathbb{R}^{d+1}} \frac{d^{d+1}k}{(2\pi)^{d+1}} f(k) e^{-ikx}. \quad (\text{A.7})$$

Because this integrals are appearing frequently the abbreviations

$$\int_x = \int_{\mathbb{R}^{d+1}} d^{d+1}x \quad (\text{A.8})$$

$$\int_k = \int_{\mathbb{R}^{d+1}} \frac{d^{d+1}k}{(2\pi)^{d+1}} \quad (\text{A.9})$$

as well as

$$\begin{aligned} \int_{x^0} &= \int dx^0 \\ \int_{\mathbf{x}} &= \int d^d \mathbf{x} \\ \int_{k^i} &= \int \frac{dk^i}{2\pi} \quad i = 0 \dots d \end{aligned} \quad (\text{A.10})$$

are used, where also other symbols representing spacetime points or momentum space points might be used.

## A. Notations and Conventions

**Indices** Indices indicate that we only consider a single component, which usually is a scalar. For example does  $z_a$  represent the  $a_{th}$  component of the vector  $z$ . Sometimes the vector  $z$  is also denoted as

$$z = (z_a)_{a=1\dots N} \text{ or } z = (z_a), \quad (\text{A.11})$$

if the second notation is not ambiguous. We use the Einstein sum convention, where over indices appearing twice is summed:

$$\sum_a x_a x_a = x_a x_a \quad (\text{A.12})$$

and say that the index  $a$  is contracted. We don't distinguish between upper and lower case indices. Sometimes it is convenient to write them on different levels to improve readability. Definitions that are given in index notation are definitions that should hold for all free non contracted indices. We usually omit the notation "for  $a = 1 \dots N$ ". The symbol  $\dagger$  represents the hermitian conjugate of a vector or matrix. If it stands next to a scalar it represents complex conjugation, for example

$$z_a^\dagger = z_a^* \quad (\text{A.13})$$

**Derivatives** At some point many derivatives appear which would make the notation hard to read. We thus use the following notations: Derivatives act on the symbol directly to the right, if this symbol represents a classical or quantum field.

$$\partial_i f g = (\partial_i f) g \quad (\text{A.14})$$

If derivatives stand in front of a mathematical operator the whole term included by the operator is derived, e.g.

$$\partial_i \int f g = \partial_i \left( \int f g \right). \quad (\text{A.15})$$

The  $\nabla$  Operator operates on scalars and gives a column vector of partial derivatives. If the Nabla Operator is multiplied with a vector of equal length, this multiplication should be understood as the standard euclidean dot product, e.g.

$$\begin{aligned} \nabla z_a^\dagger \nabla z_a &= (\nabla z_a^\dagger) (\nabla z_a) \\ &= \sum_a \nabla z_a^\dagger \nabla z_a = \sum_a \sum_i \partial_i z_a^\dagger \partial_i z_a \\ &= \partial_i z_a^\dagger \partial_i z_a \end{aligned} \quad (\text{A.16})$$

**Functional calculus** A functional of some function  $\phi$  is written with squared brackets, e.g.  $S[\phi]$ . The functional derivative, defined analogously to a finite dimensional derivative, is denoted as

$$\frac{\delta}{\delta \phi(x)} \quad (\text{A.17})$$

Sometimes the argument is omitted. It should be remembered though, as  $\phi(x)$  and  $\phi(y)$  are independent degrees of freedom for  $x \neq y$ . We for example have:

$$\frac{\delta}{\delta \phi(x)} \phi(y) = \delta(x - y). \quad (\text{A.18})$$

Conversely the functional multiplication is usually defined with an integral, which is sometimes abbreviated by a  $\cdot$  and sometimes even omitted. This is also true for other dot products.

$$\phi \cdot J := \int \phi(x) J(x) \quad (\text{A.19})$$

**Path integral measure** The measure of the path integral is denoted by a curly capital  $\mathcal{D}$ . For example

$$Z[J] := \int \mathcal{D}\varphi_a \exp(iS[\varphi] + i\varphi \cdot J) \quad (\text{A.20})$$

For details on the definition of the path integral see for example [12].



## B. Mathematical identities

### B.1. Proof that $U(N)/U(N-1) = S^{2N-1}$

Let's first take a point on the sphere, for example the  $2N$  component vector

$$z_0 = (1, 0, \dots, 0)^T. \quad (\text{B.1})$$

We have a natural action of the unitary group on the complex space defined by matrix multiplication. This action acts transitively on the sphere, which means that

$$\{z \in \mathbb{C} \mid \exists U \in U(N), z = Uz_0\} = U(N)z_0 = S^{2N-1}. \quad (\text{B.2})$$

We already saw in (2.10), where we analyzed the symmetry of the ground state, that  $z_0$  stays invariant under the action of  $U(N-1)$ . It is possible to show that this property is independent of  $z_0$ . So we have

$$U(N)z_0 = U(N)U(N-1)z_0 = S^{2N-1} \quad (\text{B.3})$$

or for any  $z \in S^{2N-1}$  there exists a  $V \in U(N)$  with

$$x = VU(N-1)z_0. \quad (\text{B.4})$$

This is equivalent to the fact that the sphere  $S^{2N-1}$  consists of right cosets  $VU(N-1)$ , which lie in the coset space  $U(N)/U(N-1)$ . Alternatively we can note that for an arbitrary element  $U \in U(N)$  with representation

$$U = \begin{pmatrix} u_{11} & u_{21} & \dots & u_{N1} \\ u_{12} & & & \\ \vdots & & W & \\ u_{1N}^\dagger & & & \end{pmatrix} \quad (\text{B.5})$$

we have

$$(U^\dagger U)_{11} = u_{11}^2 + u_{12}^\dagger u_{12} + \dots + u_{1N}^\dagger u_{1N} = 1. \quad (\text{B.6})$$

Using this one can convince himself that the map

$$U(N) \rightarrow S^{2N-1} \subseteq \mathbb{R}^{2N}, \quad U \mapsto (\Re(u_{11}), \Im(u_{11}), \Re(u_{12}), \Im(u_{12}), \dots, \Re(u_{1N}), \Im(u_{1N})) \quad (\text{B.7})$$

is continuous, surjective and that  $U(N-1)$  is mapped to a single point. This again allows for the construction of a homeomorphism  $U(N)/U(N-1) \cong S^{2N-1}$ .

### B.2. The Gaussian integral

In the computation of the effective theory we need to compute a Gaussian integral (2.50). To derive the shown expression we remember the casual Gaussian integral

$$\int dx_1 \dots dx_n e^{-\mathbf{x}^T \mathbf{x}} = \sqrt{\pi^n}. \quad (\text{B.8})$$

A slight generalization can be obtained for a symmetric positive definite matrix  $A$ . If  $A$  is symmetric,  $A$  is diagonalized in an orthonormal basis, so by an orthonormal transformation  $O$ . Using the transformation  $x \rightarrow Ox$  we obtain immediatly

$$\int dx_1 \dots dx_n e^{-\mathbf{x}^T A \mathbf{x}} = \sqrt{\frac{\pi^n}{\det(A)}}, \quad (\text{B.9})$$

## B. Mathematical identities

where the positive definiteness of  $A$  guarantees that the integral is well defined. This formula is used in our case in the context of the integral

$$\int dx_1 \dots dx_n e^{-\frac{i}{2} \mathbf{x}^T A \mathbf{x} + i \mathbf{J}^T \mathbf{x}} . \quad (\text{B.10})$$

Completing the squares gives

$$-\frac{i}{2} \mathbf{x}^T A \mathbf{x} + i \mathbf{J}^T \mathbf{x} = -\frac{i}{2} [(\mathbf{x}^T - \mathbf{J}^T A^{-1}) A (\mathbf{x} - A^{-1} \mathbf{J}) - \mathbf{J}^T A^{-1} \mathbf{J}] . \quad (\text{B.11})$$

With the reparametrization  $x \rightarrow x - A^{-1} \mathbf{J}$  we obtain

$$\int dx_1 \dots dx_n e^{-\frac{i}{2} \mathbf{x}^T A \mathbf{x} + i \mathbf{J}^T \mathbf{x}} = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{i}{2} \mathbf{J}^T A^{-1} \mathbf{J}} . \quad (\text{B.12})$$

The turn of the integration variables into the complex plane is justified because the integrand vanishes at infinity. The integral can be used in Quantum Field Theory in the shape of the path integral. The determinant of  $A$  might then diverge, but cancels out in all expectation values. The physically relevant factor is the exponential containing the kernel  $A^{-1}$ , which in our case will be the greens function of a differential operator.

## C. Effective Theory

### C.1. Effective action in Schwinger Keldysh formalism

In the derivation of the low energy effective theory the effective action is obtained from an integration over the longitudinal mode  $\delta\rho$ . However we neglected the fact that for the study of non equilibrium quantum field theory integrations have to be extended to the Schwinger-Keldysh contour and time ordering has to be understood as an ordering relation on that contour. To show that the naive integration is justified we refer to the appendix of the paper [1]. The generating functional can be represented in the presence of sources via the functional-integral form

$$Z[J, \rho_D] = \int [d\varphi_0^+] [d\varphi_0^-] \langle \varphi_0^+ | \rho_D | \varphi_0^- \rangle \times \int_{\varphi_0^+, \mathcal{C}}^{\varphi_0^-} \mathcal{D}\varphi \exp \left( iS_{\mathcal{C}}[\varphi] + i \int_{x, \mathcal{C}} \varphi(x) J(x) \right) \quad (\text{C.1})$$

with  $S_{\mathcal{C}} = S[\varphi^+] - S[\varphi^-]$  being the action on the closed contour and

$$S[\varphi^\pm] = \int_{t_0} dx^0 \int d^d x \mathcal{L}[\varphi^\pm]. \quad (\text{C.2})$$

$\rho_D(t_0)$  is the density operator that describes the non equilibrium system at the initial time and  $|\varphi_0^\pm\rangle$  describes the set of eigenstates of the field operators at the initial time. Assume now that our system contains two types of degrees of freedom, heavy ones  $\psi$  which are suppressed compared to the light ones  $\phi$ . An operator measuring only light degrees of freedom should then fulfill

$$\int [d\varphi_0^+] [d\varphi_0^-] \langle \varphi_0^+ | \rho_D | \varphi_0^- \rangle \times \int_{\varphi_0^+, \mathcal{C}}^{\varphi_0^-} \mathcal{D}\varphi \mathcal{O}[\phi] e^{iS_{\mathcal{C}}[\varphi]} = \int [d\phi_0^+] [d\phi_0^-] \langle \phi_0^+ | \rho_D^{\text{eff}} | \phi_0^- \rangle \times \int_{\phi_0^+, \mathcal{C}}^{\phi_0^-} \mathcal{D}\phi \mathcal{O}[\phi] e^{iS_{\mathcal{C}}^{\text{eff}}[\phi]} \quad (\text{C.3})$$

By comparison we see that

$$\langle \phi_0^+ | \rho_D^{\text{eff}} | \phi_0^- \rangle e^{iS_{\mathcal{C}}^{\text{eff}}[\phi]} = \int [d\psi_0^+] [d\psi_0^-] \langle \varphi_0^+ | \rho_D | \varphi_0^- \rangle \times \int_{\varphi_0^+, \mathcal{C}}^{\varphi_0^-} \mathcal{D}\psi e^{iS_{\mathcal{C}}[\varphi]}. \quad (\text{C.4})$$

In the vicinity of a non thermal fixed point the system becomes universal and the behavior is independent of the details in the initial state. This gives a justification to neglect the initial state and to simplify (C.4) to

$$e^{iS_{\mathcal{C}}^{\text{eff}}[\phi]} = \int_{\mathcal{C}} \mathcal{D}\psi e^{iS_{\mathcal{C}}[\varphi]}. \quad (\text{C.5})$$

The effective action can now be computed on only one part of the contour, leading to a well known functional integration to obtain an effective theory.

### C.2. Derivation of the parametrized Lagrangian

The motivation for the parametrization

$$(\varphi_a)_{a=1\dots N} = \sqrt{\rho} e^{i\theta} \frac{1}{\sqrt{1+z^\dagger z}} \begin{pmatrix} 1 \\ z \end{pmatrix} \quad (\text{C.6})$$

was given in the main text. In this section we want to compute the resulting terms in the Lagrangian, where total derivative terms and boundary terms are disregarded also in cases where an equal sign is used.

**First kinetic term** Let's first compute:

$$\partial_t \varphi_a = \partial_t \left( \sqrt{\frac{\rho}{1+z^\dagger z}} \right) e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a + \sqrt{\frac{\rho}{1+z^\dagger z}} (i\partial_t \theta) e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a + \sqrt{\frac{\rho}{1+z^\dagger z}} e^{i\theta} \begin{pmatrix} 0 \\ \partial_t z \end{pmatrix}_a \quad (\text{C.7})$$

which yields

$$\varphi_a^\dagger (\partial_t \varphi_a) = \sqrt{\frac{\rho}{1+z^\dagger z}} (1+z^\dagger z) \partial_t \left( \sqrt{\frac{\rho}{1+z^\dagger z}} \right) + \frac{\rho}{1+z^\dagger z} (1+z^\dagger z) \partial_t \theta + \frac{\rho}{1+z^\dagger z} (1+z^\dagger \partial_t z). \quad (\text{C.8})$$

Note that some simplifications can appear because we consider a parametrization of a sphere, for which it holds:

$$x^\dagger(\theta, z)x(\theta, z) = 1 \quad (\text{C.9})$$

When subtracting a term from it's complex conjugate real terms are canceled.

$$\varphi_a^\dagger (\partial_t \varphi_a) - \partial_t (\varphi_a^\dagger) \varphi_a = i\rho \dot{\theta} + i\rho \dot{\theta} + \frac{\rho}{1+z^\dagger z} (1+z^\dagger \partial_t z) - \frac{\rho}{1+z^\dagger z} (1+\partial_t(z^\dagger)z) \quad (\text{C.10})$$

Finally we can compute the corresponding term of the Lagrangian.

$$\frac{i}{2} (\varphi_a^\dagger (\partial_t \varphi_a) - \partial_t (\varphi_a^\dagger) \varphi_a) = -\rho \dot{\theta} + \frac{i}{2} \frac{\rho}{1+z^\dagger z} (z^\dagger \partial_t z - \partial_t(z^\dagger)z) \quad (\text{C.11})$$

**Second kinetic term** The spatial derivative is more complicated due to less cancelations. We first take the simple spatial derivative:

$$\begin{aligned} \partial_i \varphi_a &= \frac{\partial_i \sqrt{\rho}}{\sqrt{1+z^\dagger z}} e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a - \frac{1}{2} \frac{\sqrt{\rho}}{\sqrt{1+z^\dagger z}^3} [(\partial_i z^\dagger)z + z^\dagger(\partial_i z)] e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a \\ &+ \sqrt{\frac{\rho}{1+z^\dagger z}} (i\partial_i \theta) e^{i\theta} \begin{pmatrix} 1 \\ z \end{pmatrix}_a + \sqrt{\frac{\rho}{1+z^\dagger z}} e^{i\theta} \begin{pmatrix} 0 \\ \partial_i z \end{pmatrix}_a \end{aligned} \quad (\text{C.12})$$

The product with the complex conjugate results in the following terms:

$$\begin{aligned} (\partial_i \varphi_a^\dagger) (\partial_i \varphi_a) &= (\partial_i \sqrt{\rho})^2 + \frac{1}{4} \frac{\rho}{(1+z^\dagger z)^2} \left[ (\partial_i z^\dagger)z + z^\dagger(\partial_i z) \right]^2 + \rho (\partial_i \theta)^2 + \frac{\rho}{1+z^\dagger z} (\partial_i z^\dagger) (\partial_i z) \\ &- 2 \cdot \frac{1}{2} (\partial_i \sqrt{\rho}) \frac{\sqrt{\rho}}{1+z^\dagger z} \left[ (\partial_i z^\dagger)z + z^\dagger(\partial_i z) \right] + \partial_i \sqrt{\rho} \frac{\sqrt{\rho}}{1+z^\dagger z} \left[ (\partial_i z^\dagger)z + z^\dagger(\partial_i z) \right] \\ &- \frac{1}{2} \frac{\rho}{1+z^\dagger z} \left[ (\partial_i z^\dagger)z + z^\dagger(\partial_i z) \right]^2 + i(\partial_i \theta) \frac{\rho}{1+z^\dagger z} \left[ (\partial_i z^\dagger)z - z^\dagger(\partial_i z) \right] \\ &= (\partial_i \sqrt{\rho})^2 + \rho \left\{ \frac{1}{4} \frac{1}{(1+z^\dagger z)^2} \left[ (\partial_i z^\dagger)z + z^\dagger(\partial_i z) \right]^2 + (\partial_i \theta)^2 + \frac{1}{1+z^\dagger z} (\partial_i z^\dagger) (\partial_i z) \right. \\ &\quad \left. - \frac{1}{2} \frac{1}{1+z^\dagger z} \left[ (\partial_i z^\dagger)z + z^\dagger(\partial_i z) \right]^2 + \frac{i(\partial_i \theta)}{1+z^\dagger z} \left[ (\partial_i z^\dagger)z - z^\dagger(\partial_i z) \right] \right\} \\ &= (\partial_i \sqrt{\rho})^2 + \rho \left\{ -\frac{1}{4} \frac{1}{(1+z^\dagger z)^2} \left[ (\partial_i z^\dagger)z + z^\dagger(\partial_i z) \right]^2 + (\partial_i \theta)^2 + \frac{1}{1+z^\dagger z} (\partial_i z^\dagger) (\partial_i z) \right. \\ &\quad \left. + \frac{i(\partial_i \theta)}{1+z^\dagger z} \left[ (\partial_i z^\dagger)z - z^\dagger(\partial_i z) \right] \right\} \end{aligned} \quad (\text{C.13})$$

The terms can be slightly summarized to clarify the structure of the terms. Some terms are canceled by adding a total derivative term. We finally obtain:

$$\begin{aligned} (\partial_i \varphi_a^\dagger) (\partial_i \varphi_a) &= (\partial_i \sqrt{\rho})^2 + \rho \left[ \left( \partial_i \theta + \frac{i}{2} \frac{(\partial_i z^\dagger)z - z^\dagger(\partial_i z)}{(1+z^\dagger z)^2} \right)^2 \right. \\ &\quad \left. + \frac{(\partial_i z^\dagger) (\partial_i z)}{1+z^\dagger z} - \frac{(\partial_i z^\dagger)z \cdot z^\dagger(\partial_i z)}{(1+z^\dagger z)^2} \right]. \end{aligned} \quad (\text{C.14})$$

**Parametrized Lagrangian** Using additionally that  $\varphi_a^\dagger \varphi_a = \rho$  we are now in the position to construct the Lagrangian parametrized by the  $\theta$  and  $z$  fields.

$$\mathcal{L} = -\rho\dot{\theta} + \frac{i}{2} \frac{\rho}{1+z^\dagger z} (z^\dagger \partial_t z - \partial_t(z^\dagger)z) \quad (\text{C.15})$$

$$\begin{aligned} & - \frac{1}{2m} \left[ (\partial_i \sqrt{\rho})^2 + \rho \left\{ \left( \partial_i \theta + \frac{i}{2} \frac{1}{(1+z^\dagger z)^2} \left[ (\partial_i z^\dagger)z - z^\dagger (\partial_i z) \right] \right)^2 \right. \right. \\ & \left. \left. + \frac{1}{1+z^\dagger z} (\partial_i z^\dagger)(\partial_i z) - \frac{1}{(1+z^\dagger z)^2} \left[ (\partial_i z^\dagger)z \cdot z^\dagger (\partial_i z) \right] \right\} \right] + \frac{g}{2} \rho^2 - \mu \rho \end{aligned} \quad (\text{C.16})$$

Here we need to consider that  $\rho$  itself shouldn't be understood as a field, because the vacuum state is realized for  $\rho = \rho_0 \neq 0$ .

### C.3. Derivation of coupling functions

Next we shall understand the scaling of the vertices. For this we take a closer look at the terms coming from the effective action. The precise form is not that relevant for this thesis, as we will only use the scaling of one vertex to lowest order, which can directly be read of the action. However for future works it might be useful to have an insight in the form of coupling functions to derive Feynman rules and to generalize the scaling analysis. We will now approximate the effective action to second order in spatial derivatives, and to quartic order in fields. In the following in all derivatives only refer to the field directly following the derivative to simplify the notation.

In the following all terms are approximated to quartic order in derivatives and quartic order in fields. Let's first compute:

$$\begin{aligned} \left( \nabla \theta + \frac{i}{2} \frac{z^\dagger \nabla z - \nabla z^\dagger z}{1+z^\dagger z} \right)^2 &= (\nabla \theta)^2 + i \nabla \theta \frac{z^\dagger \nabla z - \nabla z^\dagger z}{1+z^\dagger z} \\ &\quad - \frac{1}{4} \frac{(z^\dagger \nabla z)^2 - (z^\dagger \nabla z)(\nabla z^\dagger z) - (\nabla z^\dagger z)(z^\dagger \nabla z) + (\nabla z^\dagger z)^2}{1+z^\dagger z} \\ &\approx (\nabla \theta)^2 + i \nabla \theta \left[ z^\dagger \nabla z - \nabla z^\dagger z \right] \\ &\quad - \frac{1}{4} \left[ (z^\dagger \nabla z)^2 - (z^\dagger \nabla z)(\nabla z^\dagger z) - (\nabla z^\dagger z)(z^\dagger \nabla z) + (\nabla z^\dagger z)^2 \right] \end{aligned} \quad (\text{C.17})$$

This means for  $J$  at the order we are investigating, using  $\frac{1}{1+z^\dagger z} = 1 - z^\dagger z + \text{h.o.t.}$

$$\begin{aligned} J(x) &= -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) - \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) z^\dagger z \\ &\quad - \frac{1}{2m} \{ (\nabla \theta)^2 + i \nabla \theta \left[ z^\dagger \nabla z - \nabla z^\dagger z \right] - \frac{1}{4} \left[ (z^\dagger \nabla z)^2 - (z^\dagger \nabla z)(\nabla z^\dagger z) - (\nabla z^\dagger z)(z^\dagger \nabla z) + (\nabla z^\dagger z)^2 \right] \\ &\quad + (\nabla z^\dagger)(\nabla z) - (\nabla z^\dagger)(\nabla z) z^\dagger z - (\nabla z^\dagger)z \cdot z^\dagger (\nabla z) \} \end{aligned} \quad (\text{C.18})$$

This yields all possible crossterms of  $J \cdot J$ . I'll use a  $\cdot$  to separate terms coming from both currents, as we later need to insert the effective coupling. However inserting it now would make things even more confusing.

$$\begin{aligned} J \cdot J &= \dot{\theta} \cdot \dot{\theta} - \dot{\theta} \cdot \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) + \dot{\theta} \cdot \frac{1}{2m} \left\{ (\nabla \theta)^2 + i \nabla \theta \left[ z^\dagger \nabla z - \nabla z^\dagger z \right] + (\nabla z^\dagger)(\nabla z) \right\} \\ &\quad + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \cdot \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) - \frac{1}{2m} \left[ (\nabla \theta)^2 + (\nabla z^\dagger)(\nabla z) \right] \right\} \\ &\quad - \frac{1}{2m} (\nabla \theta)^2 \cdot \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \quad \left( \text{without the term } -\frac{1}{2m} \left[ (\nabla \theta)^2 + (\nabla z^\dagger)(\nabla z) \right] \right) \\ &\quad + \frac{1}{2m} i \nabla \theta \left[ z^\dagger \nabla z - \nabla z^\dagger z \right] \cdot \dot{\theta} - \frac{1}{2m} (\nabla z^\dagger)(\nabla z) \cdot \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \end{aligned} \quad (\text{C.19})$$

### C. Effective Theory

This yields for  $J(x) \cdot \left(-\frac{1}{g} - \frac{\nabla_x^2}{4m\rho_0 g^2}\right) J(x)$ :

$$\begin{aligned}
\dots &= \dot{\theta} \left(-\frac{1}{g}\right) \dot{\theta} - \dot{\theta} \left(-\frac{1}{g}\right) \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) + \dot{\theta} \left(-\frac{1}{g}\right) \frac{1}{2m} \left\{ (\nabla\theta)^2 + i\nabla\theta [z^\dagger \nabla z - \nabla z^\dagger z] + (\nabla z^\dagger)(\nabla z) \right\} \\
&+ \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \left(-\frac{1}{g}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) - \frac{1}{2m} [(\nabla\theta)^2 + (\nabla z^\dagger)(\nabla z)] \right\} \\
&- \frac{1}{2m} (\nabla\theta)^2 \left(-\frac{1}{g}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \\
&+ \frac{1}{2m} i\nabla\theta [z^\dagger \nabla z - \nabla z^\dagger z] \left(-\frac{1}{g}\right) \dot{\theta} \\
&- \frac{1}{2m} (\nabla z^\dagger)(\nabla z) \left(-\frac{1}{g}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \\
&- \dot{\theta} \left(-\frac{\nabla^2}{4m\rho_0 g^2}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \\
&+ \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \left(-\frac{\nabla^2}{4m\rho_0 g^2}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \tag{C.20}
\end{aligned}$$

We finally obtain the complete Lagrangian to this order:

$$\begin{aligned}
\mathcal{L}_{eff} &= \rho_0 \left[ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) - \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) z^\dagger z - \frac{1}{2m} \left\{ (\nabla\theta)^2 + i\nabla\theta [z^\dagger \nabla z - \nabla z^\dagger z] \right. \right. \\
&- \left. \left. \frac{1}{4} \left[ (z^\dagger \nabla z)^2 - (z^\dagger \nabla z)(\nabla z^\dagger z) - (\nabla z^\dagger z)(z^\dagger \nabla z) + (\nabla z^\dagger z)^2 \right] + (\nabla z^\dagger)(\nabla z) - (\nabla z^\dagger)(\nabla z) z^\dagger z - (\nabla z^\dagger)_z \cdot z^\dagger (\nabla z) \right\} \right. \\
&- \left. \frac{1}{2} \left[ \dot{\theta} \left(-\frac{1}{g}\right) \dot{\theta} - \dot{\theta} \left(-\frac{1}{g}\right) \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) + \dot{\theta} \left(-\frac{1}{g}\right) \frac{1}{2m} \left\{ (\nabla\theta)^2 + i\nabla\theta [z^\dagger \nabla z - \nabla z^\dagger z] + (\nabla z^\dagger)(\nabla z) \right\} \right. \right. \\
&+ \left. \left. \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \left(-\frac{1}{g}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) - \frac{1}{2m} [(\nabla\theta)^2 + (\nabla z^\dagger)(\nabla z)] \right\} \right. \right. \\
&- \left. \left. \frac{1}{2m} (\nabla\theta)^2 \left(-\frac{1}{g}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} + \frac{1}{2m} i\nabla\theta [z^\dagger \nabla z - \nabla z^\dagger z] \left(-\frac{1}{g}\right) \dot{\theta} \right. \right. \\
&- \left. \left. \frac{1}{2m} (\nabla z^\dagger)(\nabla z) \left(-\frac{1}{g}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} - \dot{\theta} \left(-\frac{\nabla^2}{4m\rho_0 g^2}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \right. \right. \\
&+ \left. \left. \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \left(-\frac{\nabla^2}{4m\rho_0 g^2}\right) \left\{ -\dot{\theta} + \frac{i}{2} (z^\dagger \dot{z} - \dot{z}^\dagger z) \right\} \right] \tag{C.21}
\end{aligned}$$

Now we can collect terms. The whole Lagrangian is transformed to momentum space. This is done for each field in a product of fields, labeled by order:  $k_1, k_2, \dots$ . The derivatives transform as follows:

$$\partial_t \theta \rightarrow i\omega \theta \quad \text{and} \quad \nabla \theta \rightarrow -i\mathbf{k}\theta \tag{C.22}$$

$$\partial_t z \rightarrow i\omega z \quad \text{and} \quad \partial_t z^\dagger \rightarrow -i\omega z^\dagger \tag{C.23}$$

$$\nabla z \rightarrow -i\mathbf{k}z \quad \text{and} \quad \nabla z^\dagger \rightarrow i\mathbf{k}z^\dagger \tag{C.24}$$

Contributions to  $\alpha(k_1, k_2, k_3)\theta(k_1)\theta(k_2)\theta(k_3)$ , all contributions defined to be ingoing:

$$-\frac{i}{4mg} \omega_1 \mathbf{k}_2 \mathbf{k}_3 \times 2 \tag{C.25}$$

As we are free to do all combinations of contractions this leads to:

$$\alpha(k_1, k_2, k_3) = -\frac{i\rho_0}{k_\Xi^2} \omega_1 \mathbf{k}_2 \mathbf{k}_3 + \text{all permutations} \tag{C.26}$$

#### C.4. Derivative expressions in terms of center coordinate and relative coordinates

The contribution to  $\beta(k_1, k_2, k_3, k_4)z_a^\dagger(k_1)z_a(k_2)z_b^\dagger(k_3)z_b(k_4)$ :

$$\begin{aligned} & \rho_0(\omega_1 + \omega_2) + \frac{\rho_0}{8m}(-\mathbf{k}_2\mathbf{k}_4 - \mathbf{k}_1\mathbf{k}_2 - \mathbf{k}_1\mathbf{k}_2 - \mathbf{k}_1\mathbf{k}_3 + 4\mathbf{k}_1\mathbf{k}_2 + 4\mathbf{k}_1\mathbf{k}_2) \\ & + \frac{1}{8g}(\omega_1 + \omega_2)(\omega_3 + \omega_4) - \frac{1}{8g} \frac{(\mathbf{k}_3 - \mathbf{k}_4)^2}{2k_\Xi^2}(\omega_1 + \omega_2)(\omega_3 + \omega_4) \\ & + \frac{1}{8gm}(\omega_1 + \omega_2)\mathbf{k}_3\mathbf{k}_4 \times 2 \end{aligned} \quad (\text{C.27})$$

For the coupling we are free to interchange  $1 \leftrightarrow 3$  and  $2 \leftrightarrow 4$ . Hence the coupling becomes:

$$\begin{aligned} \beta(k_1, k_2, k_3, k_4) &= (\omega_1 + \omega_2) \left( \rho_0 + \left[ \frac{1}{8g} - \frac{1}{8g} \frac{(\mathbf{k}_3 - \mathbf{k}_4)^2}{2k_\Xi^2} \right] (\omega_3 + \omega_4) + \frac{1}{2g} \frac{\mathbf{k}_3\mathbf{k}_4}{2m} \right) \\ &+ \frac{\mu}{m} \frac{1}{8g} (-\mathbf{k}_2\mathbf{k}_4 + 6\mathbf{k}_1\mathbf{k}_2 - \mathbf{k}_1\mathbf{k}_3) + \text{all permutations } 1 \leftrightarrow 3 \text{ and } 2 \leftrightarrow 4 \end{aligned} \quad (\text{C.28})$$

The contribution to  $\lambda(k_1, k_2, k_3)\theta(k_1)z_a^\dagger(k_2)z_a(k_3)$ :

$$\lambda(k_1, k_2, k_3) = \omega_1(\omega_2 + \omega_3) \left[ \frac{i}{2g} - \frac{i}{4g} \frac{(\mathbf{k}_3 - \mathbf{k}_4)^2}{2k_\Xi^2} + \frac{i}{4g} \frac{k_1^2}{2k_\Xi^2} \right] \quad (\text{C.29})$$

The contribution to  $\mu(k_1, k_2, k_3, k_4)\theta(k_1)\theta(k_2)z_a^\dagger(k_3)z_a(k_4)$ :

$$\mu(k_1, k_2, k_3, k_4) = \frac{1}{2gm}\omega_1\mathbf{k}_2(\mathbf{k}_3 + \mathbf{k}_4) - \frac{1}{4gm}\mathbf{k}_1\mathbf{k}_2(\omega_3 + \omega_4) + 1 \leftrightarrow 2 \quad (\text{C.30})$$

#### C.4. Derivative expressions in terms of center coordinate and relative coordinates

Reformulating the differential operators in terms of the new coordinates:

$$\partial_{x^i} = \frac{\partial X^i}{\partial x^i} \frac{\partial}{\partial X^i} + \frac{\partial s^i}{\partial x^i} \frac{\partial}{\partial s^i} = \frac{1}{2}\partial_{X^i} + \partial_{s^i} \quad (\text{C.31})$$

$$\partial_{y^i} = \frac{\partial X^i}{\partial y^i} \frac{\partial}{\partial X^i} + \frac{\partial s^i}{\partial y^i} \frac{\partial}{\partial s^i} = \frac{1}{2}\partial_{X^i} - \partial_{s^i} \quad (\text{C.32})$$

$$i\partial_{x^0} - (-i\partial_{y^0}) = i\partial_{X^0} \quad (\text{C.33})$$

$$\partial_{x^0}^2 - \partial_{y^0}^2 = 2\partial_{X^0}\partial_{s^0} \quad (\text{C.34})$$

$$\nabla_x^2 - \nabla_y^2 = 2\nabla_X \cdot \nabla_s \quad (\text{C.35})$$

$$\nabla_x^2 \partial_{x^0}^2 - \nabla_y^2 \partial_{y^0}^2 = 2 \frac{\partial}{\partial X^0} \frac{\partial}{\partial s^0} \left( \frac{1}{4} \nabla_X^2 + \nabla_s^2 \right) + 2\nabla_X \nabla_s \left( \frac{1}{4} \frac{\partial}{\partial X^0}^2 + \frac{\partial}{\partial s^0}^2 \right) \quad (\text{C.36})$$





## Acknowledgements

I want to thank my supervisor *Prof. Dr. Gasenzer* for integrating me in his group and for giving me the opportunity to work close to research, which was an exciting experience. I'm especially thankful for his encouragement throughout this months. He supported me to follow the presented approach and contributed many ideas of the deeper physical meaning behind the effective theory; moreover he encouraged me in my consideration how to proceed with my degree.

I would like to thank *Aleksandr N. Mikheev* for supervising the thesis to a large extent while still being a Masters-degree student himself. He was always open to make suggestions and give answers to questions, which lead to almost weekly extensive discussions. His great pool of ideas and literature was often helpful to find the missing link to the next step.

I thank *Philipp Frey* for taking time to discuss with me towards the end of my thesis when he was one of the few people spending the holiday month at university. His advice was especially helpful when I got desperate with section 3.3.4.

I'm grateful for the group with all members being very kind towards me and for many interesting discussions including a lot of distraction.

Finally I want to thank my family and friends for moral support during my entire degree and for proof reading parts of the thesis.



## Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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Ort, Datum

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Unterschrift



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